Computing approximate Nash equilibria

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Declarations

I have not submitted any work presented in this dissertation for any previous degree, or a degree at another university. All the work has been conducted during my period of study at the University of Warwick. The results presented in this dissertation have been a result of collaborative work with my PhD supervisors Artur Czumaj and Marcin Jurdziński, and I have made major and fundamental contributions to all of them. Preliminary versions of the work have been published or have been accepted for publication: parts of the results in Chapter 2 and 3 have been accepted for publication in AAMAS 2017 [14, 15], the results in Section 2.5 and in parts of the Introduction have been published in SAGT 2014 [16], the results in Chapter 4 have been published in IJCAI 2015 [17], and the results in Chapter 5 have been published in AAMAS 2016 [18]. Some results presented in section 2.2 and Chapter 6 form a part of the paper published in WINE 2016 [13], which is the result of collaborative work with Artur Czumaj, Argyrios Deligkas, John Fearnley, Marcin Jurdziński, and Rahul Savani.

Abstract

The problem of finding equilibria in non-cooperative games and understanding their properties is a central problem in modern game theory. After John Nash [45] proved that every finite game has at least one equilibrium (so-called Nash equilibrium), the natural question arose whether we can compute one efficiently. After several years of extensive research, we now know that the problem of finding a Nash equilibrium is PPAD-complete even for two-player normal-form games [10] (see also [21]), making the task of finding *approximate* Nash equilibria one of the central questions in the area of equilibrium computation. In this thesis our main goal is a new study of the complexity of various variants of the approximate Nash equilibrium. Specifically, we study algorithms for additive approximate Nash equilibria in bimatrix and multi-player games. Then, we study algorithms for relative approximate Nash equilibria in multi-player games. Furthermore, we study algorithms for optimal approximate Nash equilibria in bimatrix games and finally we study the communication complexity of additive approximate Nash equilibria in bimatrix games.

Contents

Chapter 1

Introduction

John Nash proved that every *finite game* (finite number of players and finite number of strategies) has at least one Nash equilibrium [45]. This is a guarantee that every finite game has at least one strategy profile that no player has any incentive to deviate from. But, can we compute a Nash equilibrium efficiently?

The problem of computing Nash equilibria is one of the most fundamental problems in algorithmic game theory. There were a lot of attempts to find a polynomial-time algorithm for computing Nash equilibria, with the most notable one the Lemke-Howson algorithm [41], a method to find a Nash equilibrium for two-player games. Unfortunately, it was proved that this algorithm has exponential worst-case performance [50]. Furthermore, it is now known that the problem of computing a Nash equilibrium is PPAD-complete [21], even for two-player games [10]. Given this evidence of intractability of the problem, further research has focused on the computation of *approximate* Nash equilibria. In this context—and assuming that all payoffs are normalized to be in the $[0, 1]$ interval[1]—the standard notions of

[1]It is easy to see that the set of the Nash equilibria of normal-form games is invariant under additive and positive multiplicative transformations of the payoff matrices.

approximation are the additive or relative approximation with a parameter $\varepsilon \in [0, 1]$. There are two different variants of additive/relative approximation of Nash equilibria: the *ε-Nash equilibrium* and the *ε-well-supported Nash equilibrium.*

An additive ε-Nash equilibrium is a strategy profile—one strategy for each player—in which no player can improve her payoff by more than ε through unilateral deviation from her strategy in the strategy profile. Several polynomial-time algorithms have been proposed to find additive ε-Nash equilibria in bimatrix games for $\varepsilon = 3/4$ by Kontogiannis et al. [37], for $\varepsilon = 1/2$ and for $\varepsilon = (3 - \sqrt{5})/2 \approx 0.382$ by Daskalakis et al. [23, 22], for $\varepsilon = 1/2 - 1/(3\sqrt{6}) \approx 0.364$ by Bosse et al. [7], and finally for $\varepsilon \approx 0.3393$ by Tsaknakis and Spirakis [51]. Also, a polynomial-time algorithm for additive $(1/3 + \delta)$-Nash equilibria, for any $\delta > 0$, was given for symmetric bimatrix games by Kontogiannis and Spirakis [39]. For more than two players, there is a polynomial-time algorithm to give additive $\left(\frac{m-1}{m}\right)$-Nash equilibria, where m is the number of players, and a recursive method to give an additive $\left(\frac{1}{2-\alpha}\right)$-Nash equilibrium for m players if we can find an additive α-Nash equilibrium for $m-1$ players (see [7, 9, 35]). Furthermore, Deligkas et al. gave a polynomial-time algorithm for computing additive $(1/2+\delta)$-Nash equilibria for polymatrix games [25], for any $\delta > 0$. It is also known the existence of additive ε-Nash equilibria in normal-form games with support of size $\mathcal{O}((\log m + \log n - \log \varepsilon)/\varepsilon^2)$ for arbitrarily small $\varepsilon > 0$ [42, 35, 5], where n is the number of pure strategies. This implies a quasi-polynomial-time algorithm for games with constant number of players. For lower bounds, by Feder et al. [28] it is known that, for any $\varepsilon < 1/2$, poly-logarithmic supports of strategies are needed in order to find an additive ε-Nash equilibrium in bimatrix games. In [48], Aviad Rubinstein proves that finding an additive

ε-Nash equilibrium for some constant $\varepsilon > 0$ is PPAD-complete for the classes of polymatrix and degree 3 graphical games, in which each player has only two strategies. Recently, Rubinstein [49] has provided evidence that there is a constant $\varepsilon > 0$, such that computing ε-Nash equilibria in bimatrix games in time significantly better than the algorithm of [42] may not be possible.

An additive ε-well-supported Nash equilibrium is a strategy profile in which the expected payoff of any pure strategy that is used in a mixed strategy of a player is at most ε less than the expected payoff of the best-response strategy (strategy that maximizes the expected payoff). It is a notion stronger than that of an additive ε-Nash equilibrium: every additive ε-well-supported Nash equilibrium is also an additive ε-Nash equilibrium, but not necessarily vice-versa. The smallest ε for which a polynomial-time algorithm is currently known that computes an additive ε-well-supported Nash equilibrium in an arbitrary bimatrix game is 0.6528 [38, 26, 27, 13]. By Kontogiannis and Spirakis [38] we know that one can find additive 1/2-well-supported Nash equilibria for the special class of win-lose bimatrix games in polynomial-time. Also, by Czumaj et al. [16] we know how to find an additive $(1/2+\delta)$-well-supported Nash equilibrium in symmetric bimatrix games, for any $\delta > 0$, in polynomial-time. Furthermore, the quasi-polynomial-time algorithm for additive ε-Nash equilibria is also applied to additive ε-well-supported Nash equilibria in bimatrix games, so it is known how to find additive ε-well-supported Nash equilibria in quasi-polynomial time $n^{\mathcal{O}(\log n/\varepsilon^2)}$ for arbitrarily small $\varepsilon > 0$ [38]. On the negative side, poly-logarithmic supports of strategies are required for additive ε-well-supported Nash equilibria [3, 2], for any $\varepsilon < 1$.

Our knowledge for relative approximations is limited. While the notion of the additive approximate Nash equilibria has been studied extensively in the past, significantly less attention has been paid to the notion of the relative approximate Nash equilibria: relative ε-*Nash equilibria* and relative ε-*well-supported Nash equilibria*. Any relative approximate Nash equilibrium is also an additive approximate Nash equilibrium, but not vice-versa. A relative ε-Nash equilibrium is a strategy profile in which the payoff of each player is at least $(1-\varepsilon)$ times the payoff of the best-response strategy. Most of the relevant results we are aware of appeared in the paper of Feder et al. [28] for bimatrix games. On the positive side, Feder et al. [28, Theorem 3] give a polynomial-time algorithm that finds a relative ε-Nash equilibrium for ε slightly smaller than $\frac{1}{2}$. (There exists a function $f(n) = (2 + o(1))^n$ such that for any α, $0 < \alpha < \frac{1}{8nf(n)}$, one can find in polynomial-time a relative $(\frac{1}{2} - \alpha)$-Nash equilibrium; the relative $(\frac{1}{2} - \alpha)$-Nash equilibrium found is a pure row strategy and a mixed column strategy.) On the negative side, Feder et al. [28, Theorem 1] show that for any α, $0 < \alpha \leq \frac{1}{2}$, if one limits the column player to strategies with support of size less than $\frac{\log_2 n}{\log_2(1/\alpha)}$, then it is not possible to find a relative $(\frac{1}{2} - \alpha)$-Nash equilibrium, even for constant-sum 0/1 games. Furthermore, Feder et al. [28, Theorem 2] show bimatrix constant-sum 0/1 games for which for any ε, $0 < \varepsilon < \frac{1}{2}$, no pair of mixed strategies with supports of size smaller than $\mathcal{O}(\varepsilon^{-2} \log n)$ has a relative ε-Nash equilibrium. Further, in a related work, Daskalakis [20] considers the notion of a relative ε-well-supported Nash equilibrium, a strategy profile in which any pure strategy that is played by the players has payoff at least $(1-\varepsilon)$ times the best-response payoff. He shows that the problem of finding a relative ε-well-supported Nash equilibrium in two-player games, with payoff values in $[-1, 1]$, is PPAD-complete, even for constant values of approximation in some bimatrix games. Finally, in [48] Aviad Rubinstein extends this result and he proves that finding a relative ε-well-supported Nash equilibrium in a bimatrix game with positive payoffs is PPAD-complete for constant values of approximation.

1.1 New contributions

In this thesis our main goal is a new study of the complexity of various variants of the approximate Nash equilibrium. In Chapter 2, we study polynomial-time algorithms for additive approximate Nash equilibria in bimatrix and multi-player games. Specifically, we present new techniques based on zero-sum games and their applications to additive approximate Nash equilibria in bimatrix games. Then, we present a polynomial-time algorithm to compute additive $\frac{2}{3}$-well-supported Nash equilibria in bimatrix games and a polynomial-time algorithm to compute additive $(\frac{1}{2}+\delta)$-well-supported Nash equilibria in symmetric bimatrix games, for any $\delta > 0$. At the end of this chapter, we study polynomial-time algorithms of additive approximate Nash equilibria in multi-player games. Results presented in this chapter are also presented in [13, 16, 14, 15].

In Chapter 3, we investigate algorithms for relative approximate Nash equilibria. We first present a polynomial-time algorithm for computing relative 1/2-Nash equilibria in bimatrix games and then we generalize this result to multi-player games. Results of Chapter 3 are also presented in [14].

In Chapter 4, we study polynomial-time algorithms for computing additive ε-Nash equilibria in bimatrix games that are also close to the social welfare of the game. Results of Chapter 4 are also presented in [17].

In Chapter 5, we study polynomial-time algorithms for computing additive ε-Nash equilibria in bimatrix games that are also close to variants of the social welfare of any Nash equilibrium of the game. Results of Chapter 5 are also presented in [18].

Finally, in Chapter 6 we study the communication complexity of additive approximate Nash equilibria in bimatrix games. We give an algorithm to compute an additive $(0.382 + \delta)$-Nash equilibrium and an algorithm to compute an additive $(2/3 + \delta)$-well-supported Nash equilibrium in bimatrix games under the communication complexity constraint for any $\delta > 0$. Some parts of this chapter are also presented in [13].

1.2 Definitions — bimatrix games

We consider bimatrix games (R, C), where $R, C \in [0, 1]^{n \times n}$ are the matrices of payoffs for the two players: the row player and the column player, respectively. If the row player uses a strategy i, $1 \leq i \leq n$, and if the column one uses a strategy j, $1 \leq j \leq n$, then the row player receives payoff R_{ij} and the column player receives payoff C_{ij}.

Let $\Delta = \{x \in [0, 1]^n \,:\, \sum_{i=1}^{n} x(i) = 1\}$ be the set of mixed strategies: a *mixed strategy* x is a probability distribution on the set of *pure strategies* $\{1, 2, \ldots, n\}$. If the row player uses a mixed strategy x and the column player uses a mixed strategy y, then the row player receives payoff $x^T R y$ and the column player receives payoff $x^T C y$. A pair of strategies (x, y), the former for the row player and the latter for the column player, is often referred to as a strategy profile. We define the *support* $\mathrm{supp}(x)$ of a mixed strategy x to be the set of pure strategies that have positive probability in x, i.e., $\mathrm{supp}(x) = \{i \,:\, 1 \leq i \leq n \text{ and } x(i) > 0\}$. Let e_i be the mixed strategy such that a player plays the pure strategy i, in other words e_i is the column vector that has 1 in the coordinate i and 0 elsewhere. We define $\mathcal{M}(Ry)$ the set of the pure best-response strategies (pure strategies that maximize the expected payoff) of the row player against the strategy y of the column player. Also, $\mathcal{M}(x^T C)$ is the set of the pure best-response strategies of the column player against the strategy x of the row player.

For every i, $1 \leq i \leq n$, let $R_{i\bullet}$ be the row vector of the payoffs of the payoff matrix R when the row player uses the strategy i, or in other words $e_i^T R$. Note that if the row player uses a pure strategy i, $1 \leq i \leq n$, and if the column player uses a mixed strategy y, then the row player receives payoff $R_{i\bullet}y$, or equivalently $e_i^T Ry$. Similarly, for every j, $1 \leq j \leq n$, let $C_{\bullet j}$ be the column vector of the payoffs of the matrix C when the column player uses the strategy j, or in other words Ce_j. Note that if the column player uses a pure strategy j, $1 \leq j \leq n$, and if the row player uses a mixed strategy x, then the column player receives payoff $x^T C_{\bullet j}$, or equivalently $x^T Ce_j$.

We define $f_R(x, y)$—the *row regret* of (x, y)—by $f_R(x, y) = \max(Ry) - x^T Ry$, where $\max(Ry) = \max_i(e_i^T Ry)$ and $f_C(x, y)$ —the *column regret* of (x, y)—by $f_C(x, y) = \max(x^T C) - x^T Cy$, where $\max(x^T C) = \max_j(x^T Ce_j)$. We define the maximum *regret* $f(x, y)$ of a strategy profile (x, y) by $f(x, y) = \max\{f_R(x, y), f_C(x, y)\}$. We have the following definitions.

Definition 1.1 (Nash equilibrium (NE)) *A* Nash equilibrium *is a strategy profile (x^*, y^*) such that*

- *for every i, $1 \leq i \leq n$, we have $R_{i\bullet}y^* \leq (x^*)^T Ry^*$, and*
- *for every j, $1 \leq j \leq n$, we have $(x^*)^T C_{\bullet j} \leq (x^*)^T Cy^*$,*

or, in other words, if x^* is a best-response strategy to y^* and y^* is a best-response strategy to x^*.

Proposition 1.2 (Nash equilibrium) *A strategy profile (x^*, y^*) is a* Nash equilibrium *if and only if $f(x^*, y^*) = 0$, or, in other words, the maximum regret of the players is zero.*

Definition 1.3 (Additive ε-Nash equilibrium) *For every $\varepsilon > 0$, an* additive ε-Nash equilibrium *is a strategy profile (x^*, y^*) such that*

- *for every i, $1 \leq i \leq n$, we have $R_{i\bullet}y^* - (x^*)^T Ry^* \leq \varepsilon$, and*
- *for every j, $1 \leq j \leq n$, we have $(x^*)^T C_{\bullet j} - (x^*)^T Cy^* \leq \varepsilon$,*

or, in other words, if x^* is an ε-best-response strategy to y^* and y^* is an ε-best-response strategy to x^*.

Proposition 1.4 (Additive ε-Nash equilibrium) *A strategy profile (x^*, y^*) is an* additive ε-Nash equilibrium *if and only if $f(x^*, y^*) \leq \varepsilon$, or, in other words, the maximum regret of the players is at most ε.*

Note that an additive 0-Nash equilibrium is an exact Nash equilibrium.

Definition 1.5 (Additive ε-well-supported Nash equilibrium) *For every $\varepsilon > 0$, an* additive ε-well-supported Nash equilibrium *is a strategy profile (x^*, y^*) such that*

- *for every i, $1 \leq i \leq n$, and $i' \in \mathrm{supp}(x^*)$, we have $R_{i\bullet}y^* - R_{i'\bullet}y^* \leq \varepsilon$, and*
- *for every j, $1 \leq j \leq n$, and $j' \in \mathrm{supp}(y^*)$, we have $(x^*)^T C_{\bullet j} - (x^*)^T C_{\bullet j'} \leq \varepsilon$,*

or, in other words, if every $i' \in \mathrm{supp}(x^*)$ is an ε-best-response strategy to y^* and every $j' \in \mathrm{supp}(y^*)$ is an ε-best response strategy to x^*.

Note that any *additive ε-well-supported Nash equilibrium* is an *additive ε-Nash equilibrium* and that an additive 0-well-supported Nash equilibrium is an exact Nash equilibrium.

Definition 1.6 (Relative ε-Nash equilibrium) *For every* $\varepsilon > 0$*, a* relative ε-Nash equilibrium[2] *is a strategy profile* (x^*, y^*) *such that*

- *for every* i*,* $1 \leq i \leq n$*, we have* $(x^*)^T R y^* \geq (1 - \varepsilon) \cdot R_{i\bullet} y^*$*, and*

- *for every* j*,* $1 \leq j \leq n$*, we have* $(x^*)^T C y^* \geq (1 - \varepsilon) \cdot (x^*)^T C_{\bullet j}$*.*

Note that a relative 0-Nash equilibrium is an exact Nash equilibrium.

Definition 1.7 (Relative ε-well-supported Nash equilibrium) *For every* $\varepsilon > 0$*, a* relative ε-well-supported Nash equilibrium *is a strategy profile* (x^*, y^*) *such that*

- *for every* i*,* $1 \leq i \leq n$*, and* $i' \in \text{supp}(x^*)$*, we have* $(1 - \varepsilon) \cdot R_{i\bullet} y^* \leq R_{i'\bullet} y^*$*, and*

- *for every* j*,* $1 \leq j \leq n$*, and* $j' \in \text{supp}(y^*)$*, we have* $(1 - \varepsilon) \cdot (x^*)^T C_{\bullet j} \leq (x^*)^T C_{\bullet j'}$*.*

Note that any *relative ε-well-supported Nash equilibrium* is a *relative ε-Nash equilibrium* and that a relative 0-well-supported Nash equilibrium is an exact Nash equilibrium.

Definition 1.8 (Win-lose game) *A game is* win-lose *if any payoff entry belongs in* $\{0, 1\}$*.*

Definition 1.9 (Symmetric game, symmetric Nash equilibrium) *A bimatrix game* (R, C) *is* symmetric *if* $C = R^T$*.*

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[2] Let us note that we use the definition from Daskalakis in [20] that is consistent with the notion of the additive ε-Nash equilibria, but it differs from the definition from Feder et al. [28], where one replaced the $(1 - \varepsilon)$ factor by ε. That is, the definition in Feder et al. [28] for two players was that a strategy profile (x, y) is a *"Feder-et-al."*-relative α-Nash equilibrium if $x^T R y \geq \alpha \cdot (x')^T R y$ and $x^T C y \geq \alpha \cdot x^T C y'$ for all mixed strategies x' and y'.

A symmetric Nash equilibrium *in a symmetric bimatrix game* (R, R^T) *is a strategy profile* (x^*, x^*) *such that for every* i, $1 \leq i \leq n$, *we have* $R_{i\bullet}x^* \leq (x^*)^T Rx^*$. *Note that then it also follows that for every* j, $1 \leq j \leq n$, *we have:*

$$(x^*)^T R^T_{\bullet j} = R_{j\bullet}x^* \leq (x^*)^T Rx^* = (Rx^*)^T x^* = (x^*)^T R^T x^*.$$

Let us recall a fundamental theorem of John Nash [45] about existence of symmetric Nash equilibria in symmetric games.

Theorem 1.10 (John Nash [45]) *Every finite symmetric game has a symmetric Nash equilibrium.*

Now we will describe a general lemma for the additive ε-Nash equilibria which has been used in all the previous algorithms that guarantee additive ε-Nash equilibria (see [37, 7, 22, 23, 51]).

Lemma 1.11 *Let* (x_1, y) *and* (x_2, y) *be two strategy profiles, with row regrets* $f_R(x_1, y)$, $f_R(x_2, y)$ *and column regrets* $f_C(x_1, y)$, $f_C(x_2, y)$. *Any convex combination between these two strategy profiles has regrets no larger than the convex combination of the regrets.*

Proof. Consider a convex combination $(px_1 + (1-p)x_2, y)$, for some $p \in [0, 1]$. Then, the regret of the row player is:

$$\begin{aligned} f_R(px_1 + (1-p)x_2, y) &= \max(Ry) - (px_1 + (1-p)x_2)^T Ry \\ &= \max(Ry) - p(x_1)^T Ry - (1-p)(x_2)^T Ry \\ &= p\max(Ry) + (1-p)\max(Ry) - p(x_1)^T Ry - (1-p)(x_2)^T Ry \\ &= pf_R(x_1, y) + (1-p)f_R(x_2, y), \end{aligned}$$

since $f_R(x_1, y) = \max(Ry) - (x_1)^T Ry$, and $f_R(x_2, y) = \max(Ry) - (x_2)^T Ry$.

The regret of the column player is:

$$f_C(px_1+(1-p)x_2, y) = \max((px_1+(1-p)x_2)^T C) - (px_1+(1-p)x_2)^T Cy$$
$$\leq p\max((x_1)^T C) + (1-p)\max((x_2)^T C) - p(x_1)^T Cy - (1-p)(x_2)^T Cy$$
$$= pf_C(x_1, y) + (1-p)f_C(x_2, y),$$

since $f_C(x_1, y) = \max((x_1)^T C) - (x_1)^T Cy$, and $f_C(x_2, y) = \max((x_2)^T C) - (x_2)^T Cy$. $\square$

We now give a definition that we use it to prove additive ε-well-supported Nash equilibria.

Definition 1.12 (Preventing exceeding payoffs) *We say that a strategy $x \in [0,1]^n$ for the row player* prevents exceeding $u \in [0,1]$ *if for every $j = 1, 2, \ldots, n$, we have $x^T C_{\bullet j} \leq u$ or, in other words, if the column player payoff of the best-response strategy to x does not exceed u. Similarly, we say that a strategy $y \in [0,1]^n$ for the column player* prevents exceeding $v \in [0,1]$ *if for every $i = 1, 2, \ldots, n$, we have $R_{i\bullet} y \leq v$ or, in other words, if the row player payoff of the best-response strategy to y does not exceed v.*

For brevity, we say that a strategy profile (x, y) prevents exceeding (v, u) *if x prevents exceeding u and y prevents exceeding v.*

Observe that the following system of linear constraints PE(v, u) characterizes strategy profiles (x, y) that prevent exceeding $(v, u) \in [0,1]^2$:

$$\begin{aligned}
\sum_{i=1}^{n} x(i) = 1; &\qquad x(i) \geq 0 \text{ for all } i = 1, 2, \ldots, n; \\
\sum_{j=1}^{n} y(j) = 1; &\qquad y(j) \geq 0 \text{ for all } j = 1, 2, \ldots, n; \\
R_{i\bullet} y \leq v &\qquad \text{for all } i = 1, 2, \ldots, n; \\
x^T C_{\bullet j} \leq u &\qquad \text{for all } j = 1, 2, \ldots, n.
\end{aligned}$$

Note that if (x, y) is a Nash equilibrium then, by definition, it prevents exceeding $(x^T Ry, x^T Cy)$, which implies the following Proposition.

Proposition 1.13 *If* (x, y) *is a Nash equilibrium,* $v \geq x^T Ry$*, and* $u \geq x^T Cy$*, then* $\mathrm{PE}(v, u)$ *has a solution and it prevents exceeding* (v, u)*.*

By the following proposition, in order to find an additive ε-well-supported Nash equilibrium it suffices to find a strategy profile that prevents exceeding $(\varepsilon, \varepsilon)$.

Proposition 1.14 *If a strategy profile* (x, y) *prevents exceeding* (v, u) *then it is an additive* $\max(v, u)$*-well-supported Nash equilibrium.*

Proof. Let $i' \in \mathrm{supp}(x)$ and let $i \in \{ 1, 2, \dots, n \}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq R_{i\bullet}y \leq v,$$

where the first inequality follows from $R_{i'\bullet}y \geq 0$, and the other one holds because y prevents exceeding v. Similarly, and using the assumption that x prevents exceeding u, we can argue that for all $j' \in \mathrm{supp}(y)$ and $j \in \{1, 2, \dots, n\}$, we have $x^T C_{\bullet j} - x^T C_{\bullet j'} \leq u$. It follows that (x, y) is a $\max(v, u)$-well-supported Nash equilibrium. □

We now introduce another definition which we also use to prove additive ε-well-supported Nash equilibria.

Definition 1.15 (Well supporting payoffs) *We say that a strategy* $x \in [0, 1]^n$ *for the row player* well supports $v \in [0, 1]$ *against a strategy* $y \in [0, 1]^n$ *for the column player if for every* $i \in \mathrm{supp}(x)$*, we have* $R_{i\bullet}y \geq v$*. Similarly, we say that a strategy* $y \in [0, 1]^n$ *for the column player* well supports $u \in [0, 1]$ *against a strategy* $x \in [0, 1]^n$ *for the row player if for every* $j \in \mathrm{supp}(y)$*, we have* $x^T C_{\bullet j} \geq u$*.*

For brevity, we say that a strategy profile (x, y) well supports (v, u) *if* x *well supports* v *against* y *and* y *well supports* u *against* x.

By the following proposition, in order to find an additive ε-well-supported Nash equilibrium it suffices to find a strategy profile that well supports $(1 - \varepsilon, 1 - \varepsilon)$.

Proposition 1.16 *If a strategy profile* (x, y) *well supports* (v, u) *then it is an additive* $\max(1 - v, 1 - u)$*-well-supported Nash equilibrium.*

Proof. Let $i' \in \text{supp}(x)$ and let $i \in \{1, 2, \ldots, n\}$. Then we have:

$$R_{i\bullet}y - R_{i'\bullet}y \leq R_{i\bullet}y - v \leq 1 - v,$$

where the first inequality follows from $R_{i'\bullet}y \geq v$, and the other one holds because $R_{i\bullet}y \leq 1$. Similarly, for the column player. It follows that (x, y) is a $\max(1 - v, 1 - u)$-well-supported Nash equilibrium. $\square$

We now define zero-sum bimatrix games and we present some properties of them that we use to find additive approximate Nash equilibria.

Definition 1.17 (Zero-sum game) *A bimatrix game* $(R, C) \in \mathbb{R}^{n \times n}$ *is* zero-sum *if and only if* $C = -R$.

Note that we can compute exact Nash equilibria in polynomial-time in zero-sum games using linear programming.

Let $(A, -A)$ be a zero-sum game with Nash equilibrium (x, y) and value $v = x^T Ay$. Then, by the definition of the Nash equilibrium we know that

- for every i, $1 \leq i \leq n$, we have $A_{i\bullet}y \leq v$, and
- for every j, $1 \leq j \leq n$, we have $(x)^T A_{\bullet j} \geq v$,

we can see that y prevents exceeding v. Let $(-A, A)$ be a zero-sum game with Nash equilibrium (x, y) and value $u = x^T Ay$. Then, by the definition of the Nash equilibrium we know that

- for every i, $1 \leq i \leq n$, we have $A_{i\bullet} y \geq u$, and
- for every j, $1 \leq j \leq n$, we have $(x)^T A_{\bullet j} \leq u$,

we can see that x prevents exceeding u.

1.3 Definitions — multiplayer games

Consider a normal-form game with m players. Each player has n strategies at the disposal and the entries of the payoff matrices are in $[0, 1]$. A *mixed strategy* $x \in [0, 1]^n$ is a column vector that describes a probability distribution on the n pure strategies of a player; a *support* of a mixed strategy x is the set of the pure strategies k such that $x(k) > 0$. We write x_{-i} for the strategy profile of all players except for the player i. If the player i plays a mixed strategy x_i, then the expected payoff of the player i is $u_i(x_i, x_{-i})$.

A *Nash equilibrium* is any strategy profile $x = (x_i, x_{-i})$ such that if every player randomizes according to x, then no player has an incentive to change her mixed strategy. More formally, a strategy profile $x = (x_i, x_{-i})$ is a Nash equilibrium if and only if for every i $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})$, for every mixed strategy x'_i. Equivalently, a strategy profile $x = (x_i, x_{-i})$ is a Nash equilibrium if and only if for all players i

$$u_i(x_i, x_{-i}) \geq u_i(e_k, x_{-i}) \qquad \text{for every } k = 1, \ldots, n,$$

where e_k represents the unit vector along dimension k of $\mathbb{R}^n$, that is, $e_k \in [0, 1]^n$ is the column vector with 1 in its coordinate k and 0 elsewhere.

As in bimatrix games, there are two main models of approximate Nash equilibria studied in the literature, one focusing on the *additive* quality incentive and another concerned with the *relative* quality incentive. We will describe the additive/relative ε-Nash equilibrium, assuming that the payoff matrices of the players have all entries in $[0, 1]$.

For any $\varepsilon \geq 0$, an *additive ε-Nash equilibrium* is any strategy profile $x = (x_i, x_{-i})$ such that for every player i $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}) - \varepsilon$, for every strategy x'_i. Equivalently, a strategy profile $x = (x_i, x_{-i})$ is an additive ε-Nash equilibrium if for every player i

$$u_i(x_i, x_{-i}) \geq u_i(e_k, x_{-i}) - \varepsilon \qquad \text{for every } k = 1, \ldots, n.$$

Note that an additive 0-Nash equilibrium is a Nash equilibrium.

For any $\varepsilon \geq 0$, a *relative ε-Nash equilibrium* is any strategy profile $x = (x_i, x_{-i})$ such that for every player i $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i}) - \varepsilon \cdot u_i(x'_i, x_{-i}) = (1 - \varepsilon) \cdot u_i(x'_i, x_{-i})$, for every mixed strategy x'_i. Equivalently, for any $\varepsilon \geq 0$, a strategy profile $x = (x_i, x_{-i})$ is a relative ε-Nash equilibrium if and only if for every player i

$$u_i(x_i, x_{-i}) \geq (1 - \varepsilon) \cdot u_i(e_k, x_{-i}) \qquad \text{for every } k = 1, \ldots, n.$$

Similarly, a relative 0-Nash equilibrium is a Nash equilibrium.

Part I

Algorithms for approximate Nash equilibria

Chapter 2

Additive approximate Nash equilibria

In this chapter, we study algorithms for additive approximate Nash equilibria. We first describe the state-of-art for additive ε-Nash equilibria in bimatrix games, the Tsaknakis-Spirakis algorithm [51]. Then, we present new techniques that are based on zero-sum games and their applications to additive approximate Nash equilibria in bimatrix games. After this we present methods to compute additive $\frac{2}{3}$-well-supported Nash equilibria in arbitrary bimatrix games and additive $(\frac{1}{2}+\delta)$-well-supported Nash equilibria in symmetric bimatrix games, for any $\delta > 0$. Finally, we study issues of additive approximate Nash equilibria in multi-player games.

2.1 Tsaknakis-Spirakis algorithm

In this section we will describe the Tsaknakis-Spirakis algorithm [51] for computing additive 0.3393-Nash equilibria, the state-of-art for computing additive ε-Nash equilibria in bimatrix games.

2.1.1 Directional Derivative of Regret

Our main interest in this section is to identify and study the properties of stationary points of the regret function on the set of strategy profiles, because—as observed by Tsaknakis and Spirakis [51], and as we elaborate later—they are crucial to obtain additive ε-Nash equilibria with low regret.

Let $(x, y), (x', y') \in \Delta^2$ be strategy profiles, and let f be the regret function as it was defined in the Chapter 1. We define the *directional derivative* of f at (x, y) towards (x', y') by:

$$\nabla_{(x',y')} f(x, y) = \lim_{\varepsilon \searrow 0} \frac{f\Big((1-\varepsilon)(x, y) + \varepsilon(x', y')\Big) - f(x, y)}{\varepsilon},$$

if the limit exists.

Lemma 2.1 *[51] For all strategy profiles (x, y) and (x', y'), the directional derivative $\nabla_{(x',y')} f(x, y)$ is well defined. More specifically, we have: If $f_R(x, y) > f_C(x, y)$ then*

$$\nabla_{(x',y')} f(x, y) = \nabla_{(x',y')} f_R(x, y) = \max_{\mathcal{M}(Ry)}(Ry') - x^T Ry' - (x')^T Ry + x^T Ry - f_R(x, y),$$

and if $f_R(x, y) < f_C(x, y)$ then

$$\nabla_{(x',y')} f(x, y) = \nabla_{(x',y')} f_C(x, y) = \max_{\mathcal{M}(x^T C)} \big((x')^T C\big) - (x')^T Cy - x^T Cy' + x^T Cy - f_C(x, y).$$

If $f_R(x,y) = f_C(x,y)$ *then*

$$\nabla_{(x',y')} f(x,y) = \max\Big\{\nabla_{(x',y')} f_R(x,y), \nabla_{(x',y')} f_C(x,y)\Big\}.$$

Proof. Recall that $\mathcal{M}(Ry)$ (as defined in Chapter 1) is the set of the pure best-response strategies of the row player to the mixed strategy y of the column player, and the complement of $\mathcal{M}(Ry)$ is $\overline{\mathcal{M}(Ry)} = \{1, \dots, n\} \setminus \mathcal{M}(Ry)$. Also, $\mathcal{M}(x^TC)$ is the set of the pure best-response strategies of the column player to strategy x of the row player. The complement of $\mathcal{M}(x^TC)$ is $\overline{\mathcal{M}(x^TC)} = \{1, \dots, n\} \setminus \mathcal{M}(x^TC)$. The difference of the regrets between the points $(1-\varepsilon)(x,y) + \varepsilon(x',y')$ and (x,y) is equal to

$$\begin{aligned} f\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y) = \\ = \max\Big\{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big), f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big)\Big\} \\ - f(x,y) = \max\Big\{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y), \\ f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)\Big\}. \end{aligned}$$

With this at hand and by the definition of the directional derivative of the regret function we have that the directional derivative of the regret function is equal to

$$\begin{aligned} \nabla_{(x',y')} f(x,y) = \max\Bigg\{ \lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon}, \\ \lim_{\varepsilon \searrow 0} \frac{f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} \Bigg\} \end{aligned}$$

We analyse the part

$$\lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon},$$

since the other part is analogous. First we will analyse the part $f_R\Big((1 -$

$\varepsilon)(x,y)+\varepsilon(x',y')\Big)$, so this part is equal to

$$\begin{aligned} f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) = \\ = \max\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - \Big((1-\varepsilon)x+\varepsilon x'\Big)^T R\Big((1-\varepsilon)y+\varepsilon y'\Big) \\ = \max\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - (1-\varepsilon)^2 x^T Ry - \varepsilon(1-\varepsilon)x^T Ry' \\ - \varepsilon(1-\varepsilon)(x')^T Ry - \varepsilon^2(x')^T Ry' = \max\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\ - x^T Ry - \varepsilon^2 x^T Ry + 2\varepsilon x^T Ry - \varepsilon x^T Ry' + \varepsilon^2 x^T Ry' \\ - \varepsilon(x')^T Ry + \varepsilon^2(x')^T Ry - \varepsilon^2(x')^T Ry'. \end{aligned}$$

However, we can write up the $\max\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)$ as

$$\begin{aligned} \max\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) = \\ = \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) + \max\Big\{0, \max_{\overline{\mathcal{M}(Ry)}}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\ - \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}. \end{aligned}$$

So, we have that

$$\begin{aligned} f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) = \\ = \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) + \max\Big\{0, \max_{\overline{\mathcal{M}(Ry)}}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\ - \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\} - x^T Ry - \varepsilon^2 x^T Ry + 2\varepsilon x^T Ry \\ - \varepsilon x^T Ry' + \varepsilon^2 x^T Ry' - \varepsilon(x')^T Ry + \varepsilon^2(x')^T Ry - \varepsilon^2(x')^T Ry'. \quad (2.1) \end{aligned}$$

We can see that $\lim_{\varepsilon\searrow 0}\max\Big\{0, \max_{\overline{\mathcal{M}(Ry)}}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\} = 0$, since $\lim_{\varepsilon\searrow 0}\max_{\overline{\mathcal{M}(Ry)}}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - \max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) < 0$ because $\overline{\mathcal{M}(Ry)}$ is the set of the pure strategies that they are not best-responses to y. Thus, we summarize that

$$\lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$

$$= \lim_{\varepsilon \searrow 0} \frac{\max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - x^T Ry - \varepsilon^2 x^T Ry + 2\varepsilon x^T Ry}{\varepsilon}$$
$$\frac{-\varepsilon x^T Ry' + \varepsilon^2 x^T Ry' - \varepsilon (x')^T Ry + \varepsilon^2 (x')^T Ry - \varepsilon^2 (x')^T Ry' - f(x,y)}{\varepsilon}$$
$$= \lim_{\varepsilon \searrow 0} \frac{\max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - x^T Ry + 2\varepsilon x^T Ry - \varepsilon x^T Ry'}{\varepsilon}$$
$$\frac{-\varepsilon (x')^T Ry - f(x,y)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{(1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry) + \varepsilon \max_{\mathcal{M}(Ry)}(Ry')}{\varepsilon}$$
$$\frac{-x^T Ry + 2\varepsilon x^T Ry - \varepsilon x^T Ry' - \varepsilon (x')^T Ry - f(x,y)}{\varepsilon},$$

the second equality holds since $\lim_{\varepsilon \searrow 0} \frac{-\varepsilon^2 x^T Ry + \varepsilon^2 x^T Ry' + \varepsilon^2 (x')^T Ry - \varepsilon^2 (x')^T Ry'}{\varepsilon} = 0$. The third equality holds by the property of $\max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) = (1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry) + \varepsilon \max_{\mathcal{M}(Ry)}(Ry')$, in which the equality holds since every strategy in $\mathcal{M}(Ry)$ is a pure best-response strategy to y, so has the same expected payoff against y. We have three cases: $f_R(x,y) > f_C(x,y)$, $f_R(x,y) = f_C(x,y)$, and $f_R(x,y) < f_C(x,y)$.

- If $f_R(x,y) > f_C(x,y)$,

$$\lim_{\varepsilon \searrow 0} \frac{f_C\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$
$$= \lim_{\varepsilon \searrow 0} \frac{f_C\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) - f_R(x,y)}{\varepsilon} = -\infty,$$

since $f(x,y) = f_R(x,y) > f_C(x,y)$, and

$$\lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$
$$= \max_{\mathcal{M}(Ry)}(Ry') - (x')^T Ry - x^T Ry' + x^T Ry - f_R(x,y)$$
$$= \max_{\mathcal{M}(Ry)}(Ry') - (x')^T Ry - x^T Ry' + x^T Ry - f(x,y).$$

The first equality holds since $\max_{\mathcal{M}(Ry)}(Ry) = \max(Ry)$ and the second equality holds since $f(x,y) = f_R(x,y)$.

- If $f_R(x,y) < f_C(x,y)$,

$$\lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$

$$= \lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f_C(x,y)}{\varepsilon} = -\infty,$$

since $f(x,y) = f_C(x,y) > f_R(x,y)$, and

$$\lim_{\varepsilon \searrow 0} \frac{f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$

$$= \max_{\mathcal{M}(x^TC)} ((x')^TC) - (x')^TCy - x^TCy' + x^TCy - f_C(x,y)$$

$$= \max_{\mathcal{M}(x^TC)} ((x')^TC) - (x')^TCy - x^TCy' + x^TCy - f(x,y).$$

The first equality holds since $\max_{\mathcal{M}(x^TC)}(x^TC) = \max(x^TC)$ and the second equality holds since $f(x,y) = f_C(x,y)$.

- If $f_R(x,y) = f_C(x,y)$,

$$\lim_{\varepsilon \searrow 0} \frac{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$

$$= \max_{\mathcal{M}(Ry)} (Ry') - (x')^TRy - x^TRy' + x^TRy - f_R(x,y)$$

$$= \max_{\mathcal{M}(Ry)} (Ry') - (x')^TRy - x^TRy' + x^TRy - f(x,y).$$

The first equality holds since $\max_{\mathcal{M}(Ry)}(Ry) = \max(Ry)$ and the second equality holds since $f(x,y) = f_R(x,y)$.

$$\lim_{\varepsilon \searrow 0} \frac{f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y)}{\varepsilon} =$$

$$= \max_{\mathcal{M}(x^TC)} ((x')^TC) - (x')^TCy - x^TCy' + x^TCy - f_C(x,y)$$

$$= \max_{\mathcal{M}(x^TC)} ((x')^TC) - (x')^TCy - x^TCy' + x^TCy - f(x,y).$$

The first equality holds since $\max_{\mathcal{M}(x^T C)}(x^T C) = \max(x^T C)$ and the second equality holds since $f(x, y) = f_C(x, y)$.

We summarize, so in total we have that

- If $f_R(x, y) > f_C(x, y)$,

$$\begin{aligned}
&\nabla_{(x',y')} f(x, y) = \\
&= \max\left\{ \max_{\mathcal{M}(Ry)} (Ry') - (x')^T Ry - x^T Ry' + x^T Ry - f(x, y), -\infty \right\} \\
&\qquad = \max_{\mathcal{M}(Ry)} (Ry') - (x')^T Ry - x^T Ry' + x^T Ry - f(x, y).
\end{aligned}$$

- If $f_R(x, y) < f_C(x, y)$,

$$\begin{aligned}
&\nabla_{(x',y')} f(x, y) = \\
&= \max\left\{ -\infty, \max_{\mathcal{M}(x^T C)} \Big((x')^T C\Big) - (x')^T Cy - x^T Cy' + x^T Cy - f(x, y) \right\} \\
&\qquad = \max_{\mathcal{M}(x^T C)} \Big((x')^T C\Big) - (x')^T Cy - x^T Cy' + x^T Cy - f(x, y).
\end{aligned}$$

- If $f_R(x, y) = f_C(x, y)$,

$$\begin{aligned}
&\nabla_{(x',y')} f(x, y) = \\
&= \max\left\{ \max_{\mathcal{M}(Ry)} (Ry') - (x')^T Ry - x^T Ry' + x^T Ry - f(x, y), \right. \\
&\left. \max_{\mathcal{M}(x^T C)} \Big((x')^T C\Big) - (x')^T Cy - x^T Cy' + x^T Cy - f(x, y) \right\} \\
&\qquad = \max\left\{ \max_{\mathcal{M}(Ry)} (Ry') - (x')^T Ry - x^T Ry' + x^T Ry, \right. \\
&\qquad \left. \max_{\mathcal{M}(x^T C)} \Big((x')^T C\Big) - (x')^T Cy - x^T Cy' + x^T Cy \right\} - f(x, y).
\end{aligned}$$

□

We now give a definition of the stationary and δ-stationary points of the regret function f for any $\delta \geq 0$.

Definition 2.2 *A strategy profile* (x, y) *is a stationary point of the regret function* f *if and only if for every strategy profile* (x', y') *we have* $\nabla_{(x',y')} f(x, y) \geq 0$.

Definition 2.3 *A strategy profile* (x, y) *is a* δ*-stationary point of the regret function* f *if and only if for any* $\delta \in [0, 1]$ *and for every strategy profile* (x', y') *we have* $\nabla_{(x',y')} f(x, y) \geq -\delta$.

Note that any 0-stationary point is a stationary point of the function.

Lemma 2.4 *If* (x, y) *is a stationary point of the regret function* f*, then the regrets of the players at the point* (x, y) *are equal.*

Proof. Let (x, y) be a stationary point with not equal regrets, we assume without loss of generality that $f_R(x, y) > f_C(x, y)$. We consider a direction of (x', y), where x' is a best-response strategy to y. Since, we are in a stationary point the directional derivative in this point for this direction is non-negative, $\nabla_{(x',y)} f(x, y) \geq 0$, so we have that

$$\max_{\mathcal{M}(Ry)}(Ry) - (x')^T Ry - x^T Ry + x^T Ry - f_R(x, y) \geq 0,$$

this implies that $f_R(x, y) \leq \max_{\mathcal{M}(Ry)}(Ry) - (x')^T Ry = 0$, since $(x')^T Ry = \max(Ry) = \max_{\mathcal{M}(Ry)}(Ry)$, since x' is a best-response to y. But, by definition $f_C(x, y) \geq 0$, so we have that $0 > f_C(x, y) \geq 0$, this is a contradiction. $\square$

2.1.2 An LP for Minimizing Directional Derivative

We present a linear program that—for a given strategy profile (x, y)—yields another strategy profile (x', y') for which the directional derivative of the regret function $\nabla_{(x',y')} f(x,y)$ is minimized.

$$\text{minimize } \gamma \tag{2.2}$$

$$\text{s.t. } x'(i) \geq 0, \quad y'(j) \geq 0 \qquad 1 \leq i,j \leq n \tag{2.3}$$

$$\sum_{i=1}^{n} x'(i) = 1, \quad \sum_{j=1}^{n} y'(j) = 1 \tag{2.4}$$

$$\gamma \geq R_{i\bullet} y' - (x^T R) y' - (Ry)^T x' + x^T R y \qquad i \in \mathcal{M}(Ry) \tag{2.5}$$

$$\gamma \geq C_{\bullet j}^T x' - (x^T C) y' - (Cy)^T x' + x^T C y \qquad j \in \mathcal{M}(x^T C) \tag{2.6}$$

Henceforth we refer to this linear program as *the primal.*

Consider the dual linear program, with dual variables a and b, respectively, corresponding to primal constraints (2.4); dual variables p_i, for all $i \in \mathcal{M}(Ry)$, corresponding to primal constraints (2.5); and dual variables q_j, for all $j \in \mathcal{M}(x^T C)$, corresponding to primal constraints (2.6).

$$\text{maximize } P \cdot x^T R y + Q \cdot x^T C y + a + b \tag{2.7}$$

$$\text{s.t. } p_i \geq 0 \qquad i \in \mathcal{M}(Ry) \tag{2.8}$$

$$q_j \geq 0 \qquad j \in \mathcal{M}(x^T C) \tag{2.9}$$

$$P = \sum_{i \in \mathcal{M}(Ry)} p_i, \quad Q = \sum_{j \in \mathcal{M}(x^T C)} q_j \tag{2.10}$$

$$P + Q = 1 \tag{2.11}$$

$$a \leq \sum_{i \in \mathcal{M}(Ry)} -(Ry)_k p_i + \sum_{j \in \mathcal{M}(x^T C)} [-(Cy)_k + C_{kj}] q_j \qquad 1 \leq k \leq n \tag{2.12}$$

$$b \leq \sum_{j \in \mathcal{M}(x^T C)} -(x^T C)_l q_j + \sum_{i \in \mathcal{M}(Ry)} [-(x^T R)_l + R_{il}] p_i \quad 1 \leq l \leq n \tag{2.13}$$

Henceforth, we refer to this linear program as *the dual.* Note that primal variables $x'(k)$, for all k rows, $1 \leq k \leq n$, correspond to dual constraints (2.12); primal variables $y'(l)$, for all l columns, $1 \leq l \leq n$, correspond to dual constraints (2.13); and the primal variable γ corresponds to the dual constraint (2.11). Note that the dual variables P and Q are merely auxiliary variables (defined by the auxiliary dual constraints (2.10)) that help unclutter the description of the objective function of the dual linear program, and its analysis that is to follow.

2.1.3 Intuition behind the dual

We can see that for every k the constraint

$$a \leq \sum_{i \in \mathcal{M}(Ry)} -(Ry)_k p_i + \sum_{j \in \mathcal{M}(x^T C)} [-(Cy)_k + C_{kj}] q_j,$$

can be written as

$$a \leq -P \cdot (Ry)_k - Q \cdot (Cy)_k + C_{k\bullet} q,$$

if we divide and multiply with $\sum_{j \in \mathcal{M}(x^T C)} q_j$ then we get:

$$a \leq -P \cdot (Ry)_k - Q \cdot (Cy)_k + Q \cdot C_{k\bullet} z,$$

where $z = q / \sum_{j \in \mathcal{M}(x^T C)} q_j$ is a best-response strategy to x, since $j \in \mathcal{M}(x^T C)$ and $q_j = 0$ for $j \notin \mathcal{M}(x^T C)$. It is easy to see that if $\sum_{j \in \mathcal{M}(x^T C)} q_j = 0$, we just omit the last two terms of the inequality.

Similarly, for the other constraints for every l

$$b \leq \sum_{j \in \mathcal{M}(x^T C)} -(x^T C)_l q_j + \sum_{i \in \mathcal{M}(Ry)} [-(x^T R)_l + R_{il}] p_i,$$

this implies that

$$b \leq -Q \cdot (x^T C)_l - P \cdot (x^T R)_l + p R_{\bullet l},$$

we divide and multiply with $\sum_{i \in \mathcal{M}(Ry)} p_i$ then we get

$$b \leq -Q \cdot (x^T C)_l - P \cdot (x^T R)_l + P \cdot w^T R_{\bullet l},$$

where $w = p / \sum_{i \in \mathcal{M}(Ry)} p_i$ is a best-response strategy to y, since $i \in \mathcal{M}(Ry)$ and $p_i = 0$ for $i \notin \mathcal{M}(Ry)$. It is easy to see that if $\sum_{i \in \mathcal{M}(Ry)} p_i = 0$, we just omit the last two terms of the inequality.

So, *the dual* maximizes the minimum over all rows k and all columns l. For any k, l we have

$$\begin{aligned} P \cdot w^T R_{\bullet l} - P \cdot (Ry)_k - P \cdot (x^T R)_l + P \cdot x^T Ry \\ + Q \cdot C_{k \bullet} z - Q \cdot (Cy)_k - Q \cdot (x^T C)_l + Q \cdot x^T Cy \\ \geq a + b + P \cdot x^T Ry + Q \cdot x^T Cy = \gamma. \end{aligned}$$

The inequality holds since

$$-P \cdot (Ry)_k + -Q \cdot (Cy)_k + Q \cdot C_{k \bullet} z \geq a,$$

and

$$-Q \cdot (x^T C)_l - P \cdot (x^T R)_l + P \cdot w^T R_{\bullet l} \geq b.$$

The equality holds since the value of *the primal* is equal with the value of *the dual*. But we can easily see that for any strategies x' and y' it holds that

$$\begin{aligned} P \cdot w^T R y' - P \cdot (x')^T Ry - P \cdot x^T R y' + P \cdot x^T Ry \\ + Q \cdot (x')^T Cz - Q \cdot (x')^T Cy - Q \cdot x^T C y' + Q \cdot x^T Cy \geq \gamma, \end{aligned}$$

this implies that

$$\begin{aligned} P \cdot (w^T R y' - (x')^T Ry - x^T R y' + x^T Ry) \\ + Q \cdot ((x')^T Cz - (x')^T Cy - x^T C y' + x^T Cy) \geq \gamma. \end{aligned}$$

However, from *the primal* we know that if (x, y) is δ-stationary point then

$$\nabla_{(x',y')} f(x, y) = \gamma - f(x, y) \geq -\delta,$$

or, in other words, $f(x, y) \leq \gamma + \delta$. Thus, we conclude that

$$\begin{aligned} f(x, y) \leq \gamma + \delta \leq P \cdot (w^T R y' - (x')^T R y - x^T R y' + x^T R y) \\ + Q \cdot ((x')^T C z - (x')^T C y - x^T C y' + x^T C y) + \delta, \end{aligned} \tag{2.14}$$

for any direction (x', y').

Theorem 2.5 *If P or Q is equal to zero, the* 0*-stationary point is an exact Nash equilibrium.*

Proof. We assume without loss of generality that $P = 0$. Then, by (2.14) we have that

$$f(x, y) \leq Q \cdot ((x')^T C z - (x')^T C y - x^T C y' + x^T C y). \tag{2.15}$$

Then, for the direction (x, y'), where y' is a best-response strategy to x, (2.15) becomes

$$f(x, y) \leq Q \cdot (x^T C z - x^T C y - x^T C y' + x^T C y) = Q \cdot (x^T C z - x^T C y') = 0,$$

since z is also a best-response strategy to x. □

2.1.4 Tsaknakis-Spirakis analysis of stationary points

In this subsection we will describe the analysis of Tsaknakis-Spirakis of taking approximation bound 0.3393 as it was given in [51]. Let (x, y) be a 0-stationary point. We define

$$\lambda = \min_{y' : supp(y') \subseteq \mathcal{M}(x^T C)} (w^T R y' - x^T R y') \tag{2.16}$$

and

$$\mu = \min_{x':supp(x') \subseteq \mathcal{M}(Ry)} ((x')^T Cz - (x')^T Cy). \tag{2.17}$$

From the support of x', y' we can see that y' is a best-response strategy to x and x' is a best-response strategy to y. For the direction (x, y') by (2.14) we get:

$$f(x, y) \leq P \cdot (w^T Ry' - x^T Ry') = P \cdot \lambda.$$

For the direction (x', y) by (2.14) we get:

$$f(x, y) \leq Q \cdot ((x')^T Cz - (x')^T Cy) = Q \cdot \mu.$$

Since z is a best-response strategy to x and (2.16) we get:

$$w^T Rz - x^T Rz \geq \lambda, \tag{2.18}$$

and since w is a best-response strategy to y and (2.17) we get:

$$w^T Cz - w^T Cy \geq \mu. \tag{2.19}$$

Without loss of generality we assume that $\lambda \geq \mu$, the analysis in the case $\mu \geq \lambda$ is analogous. Firstly, we analyse the regrets of the two players in the points (x, z) and (w, z). The regret of the row player in the point (x, z) is $f_R(x, z) \leq 1$, which follows by the fact that $\max(Rz) \leq 1$. The regret of the column player is $f_C(x, z) = 0$, since z is a best-response strategy to x. Now, we analyse the regrets of the players in the point (w, z). The regret of the row player is $f_R(w, z) = \max(Rz) - w^T Rz \leq 1 - \lambda - x^T Rz \leq 1 - \lambda$, the first inequality holds since $\max(Rz) \leq 1$ and (2.18). The second inequality holds since $x^T Rz \geq 0$. The regret of the column player is $f_C(w, z) = \max(w^T C) - w^T Cz \leq 1 - \mu - w^T Cy \leq 1 - \mu$, the first inequality holds since $\max(w^T C) \leq 1$ and (2.19). The second inequality holds since $w^T Cy \geq 0$.

Theorem 2.6 *The strategy profile* $\left(\frac{1}{1+\lambda-\mu} w + \frac{\lambda-\mu}{1+\lambda-\mu} x, z\right)$ *is an additive* $\left(\frac{1-\mu}{1+\lambda-\mu}\right)$*-Nash equilibrium.*

Proof. The regret of the row player, by Lemma 1.11, is:

$$
\begin{aligned}
f_R&\left(\frac{1}{1+\lambda-\mu}w+\frac{\lambda-\mu}{1+\lambda-\mu}x,z\right) = \\
&= \frac{1}{1+\lambda-\mu}f_R(w,z)+\frac{\lambda-\mu}{1+\lambda-\mu}f_R(x,z) \\
&\leq \frac{1-\lambda}{1+\lambda-\mu}+\frac{\lambda-\mu}{1+\lambda-\mu}=\frac{1-\mu}{1+\lambda-\mu}.
\end{aligned}
$$

The inequality holds since $f_R(w,z) \leq 1-\lambda$ and $f_R(x,z) \leq 1$. The regret of the column player, by Lemma 1.11, is:

$$
\begin{aligned}
f_C&\left(\frac{1}{1+\lambda-\mu}w+\frac{\lambda-\mu}{1+\lambda-\mu}x,z\right) \leq \\
&\leq \frac{1}{1+\lambda-\mu}f_C(w,z)+\frac{\lambda-\mu}{1+\lambda-\mu}f_C(x,z) \leq \frac{1-\mu}{1+\lambda-\mu}.
\end{aligned}
$$

The second inequality holds since $f_C(w,z) \leq 1-\mu$ and $f_C(x,z) = 0$. □

So, depending on the values of μ, λ we pick either the stationary point (x, y), or the strategy profile $\left(\frac{1}{1+\lambda-\mu}w+\frac{\lambda-\mu}{1+\lambda-\mu}x,z\right)$. As it was proved in [51] it gives a guaranteed bound of

$$
\max_{P,Q,\lambda,\mu}\min\left\{P\cdot\lambda, Q\cdot\mu, \frac{1-\mu}{1+\lambda-\mu}\right\} \leq 0.3393.
$$

2.1.5 The algorithm

Now, we will describe the steepest descent algorithm for finding additive $(0.3393+\delta)$-Nash equilibria, for any $\delta > 0$. The steps of the algorithm are:

1. Pick an arbitrary point.
2. Find a point (x, y), using the LP as it is described in 2.1.6, where the regrets are equal, $f_R(x, y) = f_C(x, y)$.
3. If $f(x, y) \leq 0.3393 + \delta$, stop and return (x, y).
4. Check if (x, y) is a δ-stationary point.
   - If (x, y) is a δ-stationary point, stop and if $\lambda \geq \mu$ return $(\frac{1}{1+\lambda-\mu}w + \frac{\lambda-\mu}{1+\lambda-\mu}x, z)$, else return $(w, \frac{1}{1+\mu-\lambda}z + \frac{\mu-\lambda}{1+\mu-\lambda}y)$, where w, z as defined in section 2.1.3.
5. Otherwise, find $\varepsilon_1^*, \varepsilon_2^*$ (see 2.1.7). Then, find the best direction (x', y') using *the primal* linear program, and move to a point $\Big((1-\varepsilon)(x, y) + \varepsilon(x', y')\Big)$, where $\varepsilon = \frac{\delta}{\delta + 1/\min\{\varepsilon_1^*, \varepsilon_2^*, 1/4\}}$. Go to step 2.

From our analysis in subsection 2.1.3 we can see that in the case that we are in a δ-stationary point and without loss of generality $\lambda \geq \mu$, we can bound the regret of the function as $f(x, y) \leq \lambda + \delta$, and $f(x, y) \leq \mu + \delta$. So, if we return the strategy profile $\left(\frac{1}{1+\lambda-\mu}w + \frac{\lambda-\mu}{1+\lambda-\mu}x, z\right)$ it implies that $\lambda + \delta > 0.3393 + \delta$, which implies that $\lambda > 0.3393$, and $\mu + \delta > 0.3393 + \delta$, which implies that $\mu > 0.3393$. However for this interval of values, $f\left(\frac{1}{1+\lambda-\mu}w + \frac{\lambda-\mu}{1+\lambda-\mu}x, z\right) = \frac{1-\mu}{1+\lambda-\mu} \leq 0.3393$. So, in any case our algorithm returns an additive $(0.3393 + \delta)$-Nash equilibrium.

2.1.6 How to equalize the regrets

In every step of the steepest descent algorithm we make the regrets of the players equal. Consider a strategy profile (x, y) where the regrets of the players are not equal. We assume without loss of generality that $f_R(x, y) > f_C(x, y)$. In order to equalize the regrets we propose the following linear program:

$$\begin{aligned} \text{minimize } & \max(Ry) - (x')^T Ry \\ \text{s.t. } & \max(Ry) - (x')^T Ry \geq \max\Big((x')^T C\Big) - (x')^T Cy \end{aligned}$$

The fact that the function f is continuous and the fact that there is a best-response strategy, imply that there is a strategy x', the solution of the linear program, that equalizes the two regrets. Also, we can see that $f(x', y) \leq f(x, y)$, since (x, y) is in the feasible area of the LP. The case of $f_C(x, y) > f_R(x, y)$ is analogous.

2.1.7 Running time of the descent

We will prove the following Theorem which we use to prove that the algorithm has polynomial running time.

Theorem 2.7 *If we move from a non δ-stationary (x, y) with equal regrets to a point $\Big((1-\varepsilon)(x, y) + \varepsilon(x', y')\Big)$, where (x', y') is the solution of* the primal *linear program, for any $\varepsilon \leq \min\left\{\frac{\delta}{\delta + 1/\min\{\varepsilon_1^*, \varepsilon_2^*\}}, \frac{\delta}{\delta+4}\right\}$, where $\varepsilon_1^*, \varepsilon_2^*$ are some constants $\in (0, 1]$, it holds that*

$$f\Big((1-\varepsilon)(x, y) + \varepsilon(x', y')\Big) - f(x, y) \leq \varepsilon\Big(V - f(x, y)\Big) + 2\varepsilon^2,$$

$$\begin{aligned} \textit{where } V = \max\Big\{ & V_R = \max_{\mathcal{M}(Ry)}(Ry') - (x')^T Ry - x^T Ry' + x^T Ry, \\ & V_C = \max_{\mathcal{M}(x^T C)}\Big((x')^T C\Big) - (x')^T Cy - x^T Cy' + x^T Cy\Big\}. \end{aligned}$$

Proof. By 2.1 the difference of the regrets is equal to

$$
\begin{aligned}
&f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big)-f(x,y)= \\
&\quad=\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)+\max\Big\{0,\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\
&\quad-\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}-x^TRy-\varepsilon^2x^TRy+2\varepsilon x^TRy-\varepsilon x^TRy' \\
&\qquad\qquad+\varepsilon^2x^TRy'-\varepsilon(x')^TRy+\varepsilon^2(x')^TRy-\varepsilon^2(x')^TRy'-f(x,y),
\end{aligned}
$$

this implies that

$$
\begin{aligned}
&f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big)-f(x,y)\leq \\
&\leq(1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry)+\varepsilon\max_{\mathcal{M}(Ry)}(Ry')+\max\Big\{0,\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\
&-\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}-x^TRy-\varepsilon^2x^TRy+2\varepsilon x^TRy-\varepsilon x^TRy'+\varepsilon^2x^TRy' \\
&-\varepsilon(x')^TRy+\varepsilon^2(x')^TRy-\varepsilon^2(x')^TRy'-f(x,y)=-\varepsilon\max_{\mathcal{M}(Ry)}(Ry)+\varepsilon\max_{\mathcal{M}(Ry)}(Ry') \\
&+\max\Big\{0,\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)-\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}-\varepsilon^2x^TRy \\
&\quad+2\varepsilon x^TRy-\varepsilon x^TRy'+\varepsilon^2x^TRy'-\varepsilon(x')^TRy+\varepsilon^2(x')^TRy-\varepsilon^2(x')^TRy'.
\end{aligned}
$$

The inequality holds since $\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\leq(1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry)+\varepsilon\max_{\mathcal{M}(Ry)}(Ry')$ and the equality holds since $f(x,y)=f_R(x,y)=\max(Ry)-x^TRy$ and $\max_{\mathcal{M}(Ry)}(Ry)=\max(Ry)$. Now, since $-\varepsilon^2x^TRy\leq 0$, $-\varepsilon^2(x')^TRy'\leq 0$, $x^TRy'\leq 1$, and $(x')^TRy\leq 1$ we have

$$
\begin{aligned}
&f_R\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big)-f(x,y)\leq \\
&\quad\leq-\varepsilon\max_{\mathcal{M}(Ry)}(Ry)+\varepsilon\max_{\mathcal{M}(Ry)}(Ry')+\max\Big\{0,\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\
&\quad-\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}+2\varepsilon x^TRy-\varepsilon x^TRy'-\varepsilon(x')^TRy+2\varepsilon^2 \\
&\qquad=\varepsilon\Big(V_R-f(x,y)\Big)+2\varepsilon^2+\max\Big\{0,\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\
&\qquad\qquad\qquad\qquad-\max_{\mathcal{M}(Ry)}\Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big)\Big\}.
\end{aligned}
$$

The equality holds by the definition of V_R and $f(x,y)$. We will prove that there is $\varepsilon > 0$ such that $\max_{\overline{\mathcal{M}(Ry)}} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - \max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y + \varepsilon y'\Big)\Big) < 0$. But,

$$
\begin{aligned}
&\max_{\overline{\mathcal{M}(Ry)}} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) - \max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y+\varepsilon y'\Big)\Big) \\
&\leq (1-\varepsilon)\max_{\overline{\mathcal{M}(Ry)}}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') - \max_{\mathcal{M}(Ry)}\Big(R(1-\varepsilon)y\Big) - \max_{\mathcal{M}(Ry)}\Big(R(\varepsilon y')\Big) \\
&\qquad \leq (1-\varepsilon)\max_{\overline{\mathcal{M}(Ry)}}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') - \max_{\mathcal{M}(Ry)}\Big(R(1-\varepsilon)y\Big) \\
&\qquad = (1-\varepsilon)\max_{\overline{\mathcal{M}(Ry)}}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') - (1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry). \qquad (2.20)
\end{aligned}
$$

The first inequality comes from the fact that $\max_{\overline{\mathcal{M}(Ry)}} \Big(R\Big((1-\varepsilon)y + \varepsilon y'\Big)\Big) \leq (1-\varepsilon)\max_{\overline{\mathcal{M}(Ry)}}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry')$, and $\max_{\mathcal{M}(Ry)} \Big(R\Big((1-\varepsilon)y + \varepsilon y'\Big)\Big) = \max_{\mathcal{M}(Ry)} \Big(R(1-\varepsilon)y\Big) + \max_{\mathcal{M}(Ry)} \Big(R(\varepsilon y')\Big)$, since all the strategies that belong to $\mathcal{M}(Ry)$ are best-response strategies, so they have the same expected payoff. The second inequality holds by the property that $\max_{\mathcal{M}(Ry)} \Big(R(\varepsilon y')\Big) \geq 0$. By definition we know that $\max_{\overline{\mathcal{M}(Ry)}}(Ry) < \max_{\mathcal{M}(Ry)}(Ry)$. Let $\varepsilon_1^* > 0$ be a constant such that $\max_{\overline{\mathcal{M}(Ry)}}(Ry) = (1-\varepsilon_1^*)\max_{\mathcal{M}(Ry)}(Ry)$. Then, 2.20 becomes

$$
\begin{aligned}
&(1-\varepsilon)\max_{\overline{\mathcal{M}(Ry)}}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') - (1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry) = \\
&\quad = (1-\varepsilon)(1-\varepsilon_1^*)\max_{\mathcal{M}(Ry)}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') - (1-\varepsilon)\max_{\mathcal{M}(Ry)}(Ry) \\
&\qquad = -(1-\varepsilon)\varepsilon_1^* \max_{\mathcal{M}(Ry)}(Ry) + \varepsilon \max_{\overline{\mathcal{M}(Ry)}}(Ry') \\
&\qquad\qquad < -(1-\varepsilon)\varepsilon_1^*\delta + \varepsilon,
\end{aligned}
$$

the last inequality holds since $\max_{\overline{\mathcal{M}(Ry)}}(Ry') \leq 1$ and since our algorithm has not found a $(0.3393 + \delta)$-Nash equilibrium, the regret in the point (x,y) is greater than $(0.3393 + \delta)$, so $f(x,y) > 0.3393 + \delta$, this implies that $\max_{\mathcal{M}(Ry)}(Ry) - x^T Ry > 0.3393 + \delta$, and this implies that $-\max_{\mathcal{M}(Ry)}(Ry) < -x^T Ry - 0.3393 - \delta < -\delta$, since $x^T Ry \geq 0$. This expression $-(1-\varepsilon)\varepsilon_1^*\delta + \varepsilon$ is non-positive when $\varepsilon \leq \frac{\delta}{\delta + 1/\varepsilon_1^*}$. In this case $f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y) \leq \varepsilon\Big(V_R - f(x,y)\Big) + 2\varepsilon^2$. We do the same analysis for the column player and we find another constant $\varepsilon_2^* > 0$ and we get for any $\varepsilon \leq \frac{\delta}{\delta+1/\varepsilon_2^*}$

$$f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y) \leq \varepsilon\Big(V_C - f(x,y)\Big) + 2\varepsilon^2.$$

So, there is a constant $\varepsilon^* = \min\Big\{\varepsilon_1^*, \varepsilon_2^*\Big\}$ such that we have that for any $\varepsilon \leq \min\{\frac{\delta}{\delta+1/\varepsilon^*}, \frac{\delta}{\delta+4}\}$

$$\begin{aligned} &f\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y) = \\ &= \max\Big\{f_R\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y), f_C\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) \\ &\quad - f(x,y)\Big\} \leq \max\Big\{\varepsilon\Big(V_R - f(x,y)\Big) + 2\varepsilon^2, \varepsilon\Big(V_C - f(x,y)\Big) + 2\varepsilon^2\Big\} \\ &\quad = \varepsilon(\max\Big\{V_R, V_C\Big\} - f(x,y)) + 2\varepsilon^2 = \varepsilon\Big(V - f(x,y)\Big) + 2\varepsilon^2. \quad (2.21) \end{aligned}$$

It is easy to see that if $\min\Big\{\varepsilon_1^*, \varepsilon_2^*\Big\} > 1/4$, then $\frac{\delta}{\delta+4} < \frac{\delta}{\delta + 1/\min\Big\{\varepsilon_1^*,\varepsilon_2^*\Big\}}$. So, 2.21 holds for any $\varepsilon \leq \min\Big\{\frac{\delta}{\delta+1/\min\Big\{\varepsilon_1^*,\varepsilon_2^*\Big\}}, \frac{\delta}{\delta+4}\Big\}$. □

Let $k = \frac{1}{\min\Big\{\varepsilon_1^*,\varepsilon_2^*\Big\}}$. For any $\varepsilon \leq \min\Big\{\frac{\delta}{\delta+k}, \frac{\delta}{\delta+4}\Big\}$, this implies that $\delta \geq \max\Big\{\frac{k\varepsilon}{1-\varepsilon}, \frac{4\varepsilon}{1-\varepsilon}\Big\}$. We assume without loss of generality that $k \geq 4$, so $\varepsilon \leq \frac{\delta}{\delta+k}$ and $\delta \geq \frac{k\varepsilon}{1-\varepsilon}$. However, we are not in a δ-stationary point, so $V - f(x,y) < -\delta$, it is easy to see that if $V - f(x,y) \geq -\delta$, then for any direction (x',y'), since V is the minimum in the solution of *the primal* linear program, we have $\nabla_{(x',y')} f(x,y) \geq -\delta$, so we are in a δ-stationary point and this is a contradiction. Thus,

$$\begin{aligned}
V - f(x,y) &< -\delta,\\
\Big(V - f(x,y)\Big)(\varepsilon - 1) &> k\varepsilon,\\
0 &> \Big(V - f(x,y)\Big) + k\varepsilon - \varepsilon V + \varepsilon f(x,y),\\
0 &> \varepsilon\Big(V - f(x,y)\Big) + k\varepsilon^2 - \varepsilon^2 V + \varepsilon^2 f(x,y),\\
0 &> \varepsilon\Big(V - f(x,y)\Big) + k\varepsilon^2 - 2\varepsilon^2 + \varepsilon^2 f(x,y),\\
-\varepsilon^2 f(x,y) &> \varepsilon\Big(V - f(x,y)\Big) + (k-2)\varepsilon^2 \geq \varepsilon\Big(V - f(x,y)\Big) + 2\varepsilon^2\\
&\geq f\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big) - f(x,y),
\end{aligned}$$

this implies that $(1-\varepsilon^2)f(x,y) > f\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big)$. The fifth inequality holds since $V \leq 2$ and the seventh inequality holds since $k \geq 4$. For $\varepsilon = \frac{\delta}{\delta+k}$ we have

$$\left(1 - \left(\frac{\delta}{\delta+k}\right)^2\right) f(x,y) > f\Big((1-\varepsilon)(x,y) + \varepsilon(x',y')\Big). \tag{2.22}$$

So, in every iteration by 2.22 we have

$$f(x,y) - f\Big((1-\varepsilon)(x,y)+\varepsilon(x',y')\Big) > \left(\frac{\delta}{\delta+k}\right)^2 f(x,y) \geq \left(\frac{\delta}{\delta+k}\right)^2 0.3393,$$

since we do iterations until $f(x,y) \geq 0.3393 + \delta \geq 0.3393$. But in total, we will stop when $f(x,y) \leq 0.3393 + \delta$, so we will do at most

$$\frac{1 - (0.3393+\delta)}{\left(\frac{\delta}{\delta+k}\right)^2 0.3393} < \frac{1 - 0.3393}{\left(\frac{\delta}{\delta+k}\right)^2 0.3393} \leq \mathcal{O}(1/\delta^2) \text{ number of iterations.}$$

2.2 New techniques for additive approximate Nash equilibria

In this section, we will present a new technique based on zero-sum games that gives the same approximation bound as [22, 7], equal to $\frac{3-\sqrt{5}}{2}$, for additive ε-Nash equilibria. This technique is also used in the paper [13]. Also, we give a similar technique to this technique that gives additive $\frac{1}{2}$-well-supported Nash equilibria under some conditions. Results of this section are also presented in [15, 13].

2.2.1 Additive ε-Nash equilibria

Let $(R, -R)$ and $(-C, C)$ be two zero-sum games, with $R, C \in [0,1]^{n\times n}$. Let (x^*, y^*) and $(\hat{x}, \hat{y})$ be Nash equilibria of the former zero-sum game and the latter zero-sum game, respectively. The value of the former zero-sum game is $v_R = (x^*)^T R y^*$ and for the latter is $v_C = (\hat{x})^T C \hat{y}$. We assume without loss of generality that $v_R \geq v_C$.

We are now working on the (R, C) game. Let j be a best-response strategy of the column player to the strategy x^* of the row player, and let r be a best-response strategy of the row player to the strategy j of the column player. Then, we have the following lemma.

Lemma 2.8 *The strategy profile $(\frac{1}{2-v_R}x^* + \frac{1-v_R}{2-v_R}r, j)$ is an additive $\frac{1-v_R}{2-v_R}$-Nash equilibrium for the game (R, C).*

Proof. The maximum incentive to deviate for the row player in (x^*, j) is $f_R(x^*, j) \leq 1 - v_R$, since $(x^*)^T R e_k \geq v_R$, for every $k \in \{1, \ldots, n\}$ by the Nash equilibrium definition in zero-sum games. The maximum incentive to deviate for the row player in (r, j) is $f_R(r, j) = 0$, since r is a best-response strategy to the strategy j. So, by the Lemma 1.11, the maximum regret of the row player is at most

$$\frac{1}{2-v_R} f_R(x^*, j) + \frac{1-v_R}{2-v_R} f_R(r, j) \le \frac{1-v_R}{2-v_R}.$$

The maximum incentive to deviate for the column player in (x^*, j) is $f_C(x^*, j)$ $= 0$, since j is a best-response strategy to the strategy x^*. The maximum incentive to deviate for the column player in (r, j) is $f_C(r, j) \le 1$. Therefore, by Lemma 1.11, the maximum regret of the column player is at most

$$\frac{1}{2-v_R} f_C(x^*, j) + \frac{1-v_R}{2-v_R} f_C(r, j) \le \frac{1-v_R}{2-v_R}.$$

□

Now, we will prove a result that it was also proved in [31].

Lemma 2.9 *The strategy profile $(\hat{x}, y^*)$ is an additive v_R-well-supported Nash equilibrium, and hence also an additive v_R-Nash equilibrium for the game (R, C).*

Proof. By the Nash equilibrium definition the strategy y^* prevents exceeding v_R and the strategy $\hat{x}$ prevents exceeding $v_C \le v_R$.

□

Lemma 2.8 and 2.9 allow us to give an alternative proof of the following theorem, first proved by [7, 22]. We will use this new technique for computing additive $\frac{3-\sqrt{5}}{2}$-Nash equilibria in Chapter 6.

Theorem 2.10 *For any bimatrix game (R, C) in $[0, 1]^{n \times n}$, there is a polynomial-time algorithm to compute an additive $\frac{3-\sqrt{5}}{2}$-Nash equilibrium.*

Proof. Lemma 2.8 and 2.9 give two constructions for computing additive ε-Nash equilibria in polynomial time. The function $\frac{1-v_R}{2-v_R}$ is a decreasing function of v_R and v_R is an increasing function of v_R. The two approximation bounds are equal when $v_R = \frac{3-\sqrt{5}}{2}$. Therefore, if $v_R \in [0, \frac{3-\sqrt{5}}{2}]$ return $(\hat{x}, y^*)$, else return $(\frac{1}{2-v_R}x^* + \frac{1-v_R}{2-v_R}r, j)$. This gives an additive $\frac{3-\sqrt{5}}{2}$-Nash equilibrium. $\square$

2.2.2 Additive ε-well-supported Nash equilibria

Let $(-R, R)$ and $(C, -C)$ be two zero-sum games. Let (x^*, y^*) and $(\hat{x}, \hat{y})$ be Nash equilibria of the first zero-sum game and the second zero-sum game respectively. The values of the zero-sum games are $u_R = (x^*)^T R y^*$ for the former and $u_C = (\hat{x})^T C \hat{y}$ for the latter. We assume, without loss of generality, that $u_R \geq u_C$.

Lemma 2.11 *The strategy profile $(\hat{x}, y^*)$ is an additive $(1-u_C)$-well-supported Nash equilibrium.*

Proof. The incentive of the row player to deviate from any strategy i is $1 - e_i^T R y^* \leq 1 - u_R \leq 1 - u_C$, since $e_i^T R y^* \geq u_R$ by the Nash equilibrium definition in zero-sum games, and $u_R \geq u_C$. The incentive of the column player to deviate from any strategy i is $1-\hat{x}^T C e_i \leq 1-u_C$, since $\hat{x}^T C e_i \geq u_C$ by the Nash equilibrium definition in zero-sum games. $\square$

Let (R, C) be a bimatrix game with values in $[0, 1]$. Consider two zero-sum games $(R, -R)$ with a Nash equilibrium (x_1^*, y_1^*) and value $v_R = (x_1^*)^T R y_1^*$, and $(-R, R)$ with a Nash equilibrium (x_2^*, y_2^*) and value $u_R = (x_2^*)^T R y_2^*$ for the row player. Also, consider two zero-sum games $(-C, C)$ with a Nash equilibrium $(\hat{x_1}, \hat{y_1})$ and value $v_C = \hat{x_1}^T C \hat{y_1}$, and $(C, -C)$ with a Nash equilibrium $(\hat{x_2}, \hat{y_2})$ and value $u_C = \hat{x_2}^T C \hat{y_2}$.

Lemma 2.12 *If $u_R \geq v_R$ and $u_C \geq v_C$, then we can find an additive $\frac{1}{2}$-well-supported-Nash equilibrium.*

Proof. We have four cases of analysis:

- $v_R \leq u_R \leq \frac{1}{2}$ and $v_C \leq u_C \leq \frac{1}{2}$,
- $v_R \leq u_R \leq \frac{1}{2}$ and $u_C \geq \frac{1}{2}$,
- $u_R \geq \frac{1}{2}$ and $v_C \leq u_C \leq \frac{1}{2}$,
- $u_R \geq \frac{1}{2}$ and $u_C \geq \frac{1}{2}$.

In the first case, return the strategy profile $(\hat{x_1}, y_1^*)$. For every i, $e_i^T R y_1^* \leq \frac{1}{2}$, so the incentive of the row player to deviate is at most $\frac{1}{2}$. For the column player, for every i we have $\hat{x_1}^T C e_i \leq \frac{1}{2}$, so the incentive of the column player to deviate is at most $\frac{1}{2}$. Thus, this is an additive $\frac{1}{2}$-well-supported Nash equilibrium.

In the second case, return the strategy profile $(\hat{x_2}, y_1^*)$. For every i we have $e_i^T R y_1^* \leq \frac{1}{2}$, so the incentive of the row player to deviate is at most $\frac{1}{2}$. For the column player, for every i the incentive to deviate is $1 - \hat{x_2}^T C e_i \leq \frac{1}{2}$, since by the Nash equilibrium definition for every i, we have $\hat{x_2}^T C e_i \geq u_C \geq \frac{1}{2}$. It follows that $(\hat{x_2}, y_1^*)$ is an additive $\frac{1}{2}$-well-supported Nash equilibrium.

In the third case, return the strategy profile $(\hat{x_1}, y_2^*)$. For the row player, for every i the incentive to deviate is $1 - e_i^T R y_2^* \leq \frac{1}{2}$, since by the Nash equilibrium definition for every i, we have $e_i^T R y_2^* \geq u_R \geq \frac{1}{2}$. For the column player, for every i we have that $\hat{x_1}^T C e_i \leq \frac{1}{2}$, so the incentive to deviate is at most $\frac{1}{2}$. Thus, $(\hat{x_1}, y_2^*)$ is an additive $\frac{1}{2}$-well-supported Nash equilibrium.

In the fourth case, return the strategy profile $(\hat{x_2}, y_2^*)$. For the row player, for every i the incentive to deviate is $1 - e_i^T R y_2^* \leq \frac{1}{2}$, since by the Nash equilibrium definition for every i, we have $e_i^T R y_2^* \geq u_R \geq \frac{1}{2}$. For the column player for every i the incentive to deviate is $1 - \hat{x_2}^T C e_i \leq \frac{1}{2}$, since by the Nash equilibrium definition for every i, we have $\hat{x_2}^T C e_i \geq u_C \geq \frac{1}{2}$, so $(\hat{x_2}, y_2^*)$ is an additive $\frac{1}{2}$-well-supported Nash equilibrium. $\square$

Combining conclusions of Lemma 2.8 and Lemma 2.9, we can prove the following Theorem.

Theorem 2.13 *There is a polynomial-time algorithm that for every bimatrix game with values in* $[0,1]$*, either computes an additive* $\frac{1}{3}$*-Nash equilibrium, or an additive* $\frac{1}{2}$*-well-supported Nash equilibrium.*

Proof. Let without loss of generality assume that $v_R \geq v_C$. If $v_R \geq \frac{1}{2}$, then by Lemma 2.8 we can find an additive $\frac{1-v_R}{2-v_R}$-Nash equilibrium. But, the function $\frac{1-v_R}{2-v_R}$ is a decreasing function of v_R, it achieves its maximum for $v_R = \frac{1}{2}$, where its value is $\frac{1}{3}$. If on the other hand $v_R \leq \frac{1}{2}$, by Lemma 2.9 we can find an additive $\frac{1}{2}$-well-supported Nash equilibrium. □

Lemma 2.14 *If the payoff matrix* R *is symmetric, then the values* u_R *and* v_R *are equal.*

Proof. Let (x_1^*, y_1^*) be a Nash equilibrium of the game $(R, -R)$. Then by the definition of Nash equilibrium strategy x_1^* maximizes the minimum of the expected payoff of the row player in every strategy of the column player, and strategy y_1^* minimizes the maximum of the expected payoff of the row player in every strategy of the row player. However, if (x_2^*, y_2^*) is a Nash equilibrium of the game $(-R, R)$, then by the definitions of Nash equilibrium strategy x_2^* minimizes the maximum of the expected payoff of the row player in every strategy of the column player, and strategy y_2^* maximizes the minimum of the expected payoff of the row player in every strategy of the row player. But, since the payoff matrix is symmetric, we have that

$$v_R = \max_{x \in [0,1]^n} \min_i x^T R e_i = \max_{y \in [0,1]^n} \min_i (e_i)^T R y = u_R.$$

□

Theorem 2.15 *There is a polynomial-time algorithm that computes an additive $\frac{1}{2}$-well-supported-Nash equilibrium in bimatrix games in which both payoff matrices are symmetric.*

Proof. If both payoff matrices are symmetric, then by Theorem 2.14, we have $v_R = u_R$ and $v_C = u_C$. It then follows from Lemma 2.12 that an additive $\frac{1}{2}$-well-supported Nash equilibrium can be output in polynomial time. □

2.3 Hardness results

Inspired by the previous approximation results in games with symmetric payoff matrices (see Theorem 2.15), we give some hardness results in bimatrix games where at least of one of the payoff matrices is symmetric. These results are also presented in [15].

Theorem 2.16 *Let (R, C) be a bimatrix game in which exactly one of the payoff matrices is symmetric, then computing a Nash equilibrium in this game is* PPAD*-hard.*

Proof. For the proof of this Theorem we will do a similar reduction as the reduction in [29]. Consider an arbitrary bimatrix game with payoff matrices $(A, B) \in (0, 1)^{n \times n}$.

We define a payoff matrix R such that in every position for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ we put $R_{ij} = 1$ and in every position for $i \in \{n+1, \ldots, 2n\}$ and $j \in \{n + 1, \ldots, 2n\}$ we put $R_{ij} = 0$, in every position for $i \in \{1, \ldots, n\}$ and $j \in \{n + 1, \ldots, 2n\}$ we put $R_{ij} = A_{i(j-n)}$ and in every position for $i \in \{n+1, \ldots, 2n\}$ and $j \in \{1, \ldots, n\}$ we put $R_{ij} = A^T_{(n-i)j}$. So, we have the payoff matrix

$$R = \begin{pmatrix} 1s & A \\ A^T & 0s \end{pmatrix}.$$

We define the payoff matrix C as in every position for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ we put $C_{ij} = 0$ and in every position for $i \in \{n+1, \ldots, 2n\}$ and $j \in \{n+1, \ldots, 2n\}$ we put $C_{ij} = 1$, in every position for $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$ we put $C_{ij} = B_{i(j-n)}$ and in every position for $i \in \{n+1, \ldots, 2n\}$ and $j \in \{1, \ldots, n\}$ we put $C_{ij} = 0$. So we have

$$C = \begin{pmatrix} 0s & B \\ 0s & 1s \end{pmatrix}.$$

It is easy to see that the payoff matrix R is the only symmetric payoff matrix. Let $(x_1 y_1, x_2 y_2)$ be a Nash equilibrium of the game (R, C), where x_1, x_2 are the parts of the mixed strategy probability distributions for $i, j \in \{1, \ldots, n\}$ and y_1, y_2 are the parts of the mixed strategy probability distributions for $i \in \{n+1, \ldots, 2n\}$, and $j \in \{n+1, \ldots, 2n\}$, respectively. We have that $\sum_{i=1}^{n} x_1(i) + \sum_{i=n+1}^{2n} y_1(i) = 1$ and $\sum_{j=1}^{n} x_2(j) + \sum_{j=n+1}^{2n} y_2(j) = 1$. We can see that $\sum_{i=n+1}^{2n} y_1(i) = 0$ since any pure strategy for $i \in \{1, \ldots, n\}$ is strictly dominant to any strategy in the support of y_1, since the elements of A belongs to $(0, 1)$. Also, $\sum_{j=1}^{n} x_2(j) = 0$ since any pure strategy for $j \in \{n+1, \ldots, 2n\}$ is strictly dominant to any strategy in the support of x_2, since the elements of B belongs to $(0, 1)$. Thus, any Nash equilibrium of the game (R, C) is also a Nash equilibrium of the arbitrary game (A, B). □

Theorem 2.17 *Let $(R, C) \in [0, 1]^{n \times n}$ be a bimatrix game in which both of the payoff matrices are symmetric, then computing a Nash equilibrium in this game is* PPAD*-hard.*

Proof. For the proof of this Theorem we will do a similar reduction as the reduction in Theorem 2.16. Let an arbitrary bimatrix game with payoff matrices $(A, B) \in (0,1)^{n\times n}$.

We define a payoff matrix R such that in every position for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n\}$ we put $R_{ij} = 1$ and in every position for $i \in \{n+1,\ldots,2n\}$ and $j \in \{n+1,\ldots,2n\}$ we put $R_{ij} = 0$, in every position for $i \in \{1,\ldots,n\}$ and $j \in \{n+1,\ldots,2n\}$ we put $R_{ij} = A_{i(j-n)}$ and in every position for $i \in \{n+1,\ldots,2n\}$ and $j \in \{1,\ldots,n\}$ we put $R_{ij} = A^T_{(n-i)j}$. So, we have the payoff matrix

$$R = \begin{pmatrix} 1s & A \\ A^T & 0s \end{pmatrix}.$$

We define the payoff matrix C as in every position for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n\}$ we put $C_{ij} = 0$ and in every position for $i \in \{n+1,\ldots,2n\}$ and $j \in \{n+1,\ldots,2n\}$ we put $C_{ij} = 1$, in every position for $i \in \{1,\ldots,n\}$ and $j \in \{n+1,\ldots,2n\}$ we put $C_{ij} = B_{i(j-n)}$ and in every position for $i \in \{n+1,\ldots,2n\}$ and $j \in \{1,\ldots,n\}$ we put $C_{ij} = B^T_{(i-n)j}$. So we have

$$C = \begin{pmatrix} 0s & B \\ B^T & 1s \end{pmatrix}.$$

It is easy to see that both payoff matrices R, C are symmetric. Let $(x_1 y_1, x_2 y_2)$ be a Nash equilibrium of the game (R, C), where x_1, x_2 are the parts of the mixed strategy probability distributions for $i, j \in \{1,\ldots,n\}$ and y_1, y_2 are the parts of the mixed strategy probability distributions for $i \in \{n+1,\ldots,2n\}$, and $j \in \{n+1,\ldots,2n\}$, respectively. We have that $\sum_{i=1}^{n} x_1(i) + \sum_{i=n+1}^{2n} y_1(i) = 1$ and $\sum_{j=1}^{n} x_2(j) + \sum_{j=n+1}^{2n} y_2(j) = 1$. We can see that $\sum_{i=n+1}^{2n} y_1(i) = 0$ since any pure strategy for $i \in \{1,\ldots,n\}$ is strictly dominant to any strategy in the support of y_1, since the elements of A belongs to $(0,1)$. Also,

$\sum_{j=1}^{n} x_2(j) = 0$ since any pure strategy for $j \in \{n+1, \ldots, 2n\}$ is strictly dominant to any strategy in the support of x_2, since the elements of B belongs to $(0, 1)$. Thus, any Nash equilibrium of the game (R, C) is also a Nash equilibrium of the arbitrary game (A, B). $\square$

2.4 Additive ε-well-supported NE in bimatrix games

In this section, we study polynomial-time algorithms for computing additive ε-well-supported Nash equilibria in arbitrary bimatrix games. Firstly, we give a new parametrized algorithm for computing additive 2/3-well-supported Nash equilibria in arbitrary bimatrix games and then we give a polynomial-time algorithm for computing an additive $(\frac{1}{2}+\delta)$-well-supported Nash equilibrium in the special class of symmetric bimatrix games, for any $\delta > 0$.

Our simple parametrized algorithm depends on the $\mathfrak{opt}$ value, where $\mathfrak{opt}$ is the maximum of the sum of the payoffs of the players, for computing additive ε-well-supported Nash equilibria. This algorithm inspired the work of [13] of improving the approximation bound for well-supported Nash equilibria in bimatrix games. Actually, we prove that if $\mathfrak{opt} \geq \frac{4}{3}$ then we can efficiently compute an additive $(2-\mathfrak{opt})$-well-supported Nash equilibrium with constant (size of one) support. If $\mathfrak{opt} < \frac{4}{3}$, then we can efficiently compute an additive $(\mathfrak{opt}/2)$-well-supported Nash equilibrium.

Let $(R, -R)$ and $(-C, C)$ be the two zero-sum games with Nash equilibria (x^*, y^*) and $(\hat{x}, \hat{y})$ respectively. The value of the first zero-sum game is $v_R = (x^*)^T R y^*$ and the value of the second zero-sum game is $v_C = (\hat{x})^T C \hat{y}$. We assume without loss of generality that $v_R \geq v_C$. By the definition of

Nash equilibrium we know that for every $i = 1, \ldots, n$:

$$x^{*T} R e_i \geq v_R,$$

$$e_i^T R y^* \leq v_R,$$

$$\hat{x}^T C e_i \leq v_C,$$

$$\forall j \in supp(x^*), \quad v_R = e_j^T R y^* \geq e_i^T R y^*.$$

Theorem 2.18 *Let (R,C) be a bimatrix game with values in [0,1] and* $\mathtt{opt}$ *be the maximum value of the sum* $R_{ij} + C_{ij}$ *for any pair* (i,j). *Then if* $\mathtt{opt} \geq \frac{4}{3}$, *the pure strategy profile that the sum of the entries is* $\mathtt{opt}$ *is an additive* $(2 - \mathtt{opt})$*-well-supported Nash equilibrium. Else if* $\mathtt{opt} < \frac{4}{3}$ *and* $u_R \leq \mathtt{opt}/2$ *then* $(\hat{x}, y^*)$ *is an additive* $(\mathtt{opt}/2)$*-well-supported Nash equilibrium. Else,* (x^*, y^*) *is an additive* $(\mathtt{opt}/2)$*-well-supported Nash equilibrium.*

Proof. We divide our proof in two cases:

- If $\mathtt{opt} \geq \frac{4}{3}$, then there is a strategy profile (i,j) such that $R_{ij} + C_{ij} = \mathtt{opt}$. This is an additive $(2-\mathtt{opt})$-well-supported Nash equilibrium. The maximum best-responses to the (i,j) for both players can be one so this strategy profile is an additive $\max\{1-R_{ij}, 1-C_{ij}\}$-well-supported Nash equilibrium. We assume without loss of generality that $R_{ij} \leq C_{ij}$, so $1 - R_{ij} = \max\{1 - R_{ij}, 1 - C_{ij}\} = 1 + C_{ij} - \mathtt{opt} \leq 2 - \mathtt{opt}$. So, it is an additive $(2 - \mathtt{opt})$-well-supported Nash equilibrium.

- If $\mathtt{opt} < \frac{4}{3}$, we separate the analysis in two cases :

  - If $u_R \leq \mathtt{opt}/2$, return $(\hat{x}, y^*)$. By the Nash equilibrium definition and the fact that $\mathtt{opt}/2 \geq u_R \geq u_C$, this is an additive $\mathtt{opt}/2$-well-supported Nash equilibrium.

– Else return (x^*, y^*). Then, by the Nash equilibrium definition $\forall j \in supp(x^*), \forall i$, $u_R = e_j^T R y^* \geq e_i^T R y^*$. Also,

$$C_{ij} \leq \mathtt{opt} - R_{ij} \Rightarrow x^{*T} C e_j \leq \mathtt{opt} - x^{*T} R e_j \tag{2.23}$$

But, $\forall j$, $x^{*T} R e_j \geq u_R \geq \mathtt{opt}/2$, and therefore (2.23) implies $x^{*T} C e_j \leq \mathtt{opt} - x^{*T} R e_j < \mathtt{opt}/2$. Thus, this is an additive $\mathtt{opt}/2$-well-supported Nash equilibrium.

□

2.5 Additive ε-well-supported Nash equilibria in symmetric bimatrix games

In this section we study the computation of additive ε-well-supported Nash equilibria in *symmetric* bimatrix games, a class of bimatrix games in which swapping the roles of the two players does not change the payoff matrices, that is if the payoff matrix of one is the transpose of the payoff matrix of the other. Symmetric games are an important class of games in game theory; their applications include auctions and congestion games. They have already been studied by John Nash in his seminal paper in which he introduced the concept of a Nash equilibrium; he proved that every symmetric game has at least one symmetric Nash equilibrium, that is one in which all players use the same mixed strategy [45]. Results of this section are also presented in [16].

Computing Nash equilibria in symmetric bimatrix games is known to be as hard as computing Nash equilibria in arbitrary bimatrix games because there is a polynomial-time reduction from the latter to the former [29, 46, 19]. In contrast to arbitrary bimatrix games, it is known how to compute additive $(1/3 + \delta)$-Nash equilibria in symmetric bimatrix games in polynomial-time, where $\delta > 0$ is arbitrarily small [39]. In this section we improve our understanding of the approximability of Nash equilibria in symmetric bimatrix games by considering the task of computing additive ε-well-supported Nash equilibria. Our main result is an algorithm that computes additive $(1/2 + \delta)$-well-supported Nash equilibria in symmetric bimatrix games in polynomial-time, where $\delta > 0$ is arbitrarily small (Theorem 2.21).

Our $(1/2+\delta)$-approximation algorithm splits the analysis into two cases that are then considered independently. The first case is based on the following relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, x) *prevents exceeding* $u \in [0, 1]$ if the expected payoff of every pure strategy in the symmetric game is at most u when the other player uses strategy x. This is indeed a relaxation of the concept of the symmetric Nash equilibrium because if (x^*, x^*) is a symmetric Nash equilibrium then it prevents exceeding its value (that is, the expected payoff each player gets when they both play strategy x^*). Moreover, we show that this relaxation of a symmetric Nash equilibrium is algorithmically tractable because it suffices to solve a single linear program to find a strategy profile (x, x) that prevents exceeding u, if there is one. The first case in our algorithm is to solve this linear program for $u = 1/2$ and if it succeeds then we can immediately report an additive 1/2-well-supported Nash equilibrium. Note that by the above, if there is indeed a symmetric Nash equilibrium with value 1/2 or smaller, then the linear program does have a solution.

If the first case in the algorithm fails to identify an additive 1/2-well-supported Nash equilibrium because the game has no symmetric Nash equilibrium with value 1/2 or smaller, then we consider the other, and technically more challenging case. We use another relaxation of the concept of a symmetric Nash equilibrium: we say that a strategy profile (x, y) *well supports* $u \in [0, 1]$ if the expected payoff of every pure strategy in the support of x is at least u when the other player uses strategy y, and the expected payoff of every pure strategy in the support of y is at least u when the other player uses strategy x. We observe that if a strategy profile (x, y) well supports u then it is an additive $(1 - u)$-well-supported Nash equilibrium, so in order to provide a latter it is sufficient to find a former.

Therefore, in order to obtain an additive $(1/2 + \delta)$-well-supported Nash equilibrium, we are interested in finding a strategy profile (x, y) that well supports $u \geq 1/2 - \delta$. While it may not be easy to verify if there is such a strategy profile, let alone find one, both can be achieved in polynomial-time by solving a single linear program if we happen to know the supports of strategies of each player in such a strategy profile. The obvious technical obstacle to algorithmic tractability here is that the number of all possible supports to consider is exponential in the number of pure strategies. We overcome this difficulty by proving the main technical result of the subsection (Theorem 2.19) that for every symmetric Nash equilibrium (x^*, x^*) and for every $\delta > 0$ establishes existence of a strategy profile (x, y), with both strategies having supports of constant size, that well supports $u^* - \delta$, where u^* is the value of the Nash equilibrium. Note that by the failure of the first case every symmetric Nash equilibrium has value larger than $1/2$, and hence Theorem 2.19 implies that there is such a strategy profile with constant-size supports that well supports $1/2 - \delta$. The second case of our algorithm is to solve the linear programs mentioned above for $u = 1/2 - \delta$ and for all supports I and J of sizes at most $\kappa(\delta)$—where $\kappa(\delta)$ is a constant (which depends on δ, but does not depend on the number n of pure strategies) that is specified in Theorem 2.19—and to output a solution (x, y) as soon as one is found.

In order to prove our main technical result (Theorem 2.19) we use the probabilistic method to prove existence of constant-support strategy profiles that nearly well support the expected payoffs of a Nash equilibrium. Our construction and proof are inspired by the construction of Daskalakis et al. [22] used by them to compute additive $(3-\sqrt{5})/2$-Nash equilibria in bimatrix games in polynomial-time, but our analysis is different and more involved because we need to guarantee the extra condition of nearly well supporting the equilibrium values. The general idea of using sampling and Hoeffding bounds to prove existence of approximate equilibria with small supports dates back to the papers of Althofer [1] and Lipton et al. [42], who have shown that strategies with supports of size $\mathcal{O}(\log n/\varepsilon^2)$ are necessary for additive ε-Nash equilibria in games with n strategies.

2.5.1 Computing additive ε-well-supported Nash equilibria

Fix a bimatrix game $G = (R, C)$ for the rest of the section, where $R, C \in [0,1]^{n\times n}$. We will use N to denote the number of bits needed to represent the matrices R and C with all their entries represented in binary. We say that a strategy x is *k-uniform*, for $k \in \mathbb{N}\setminus\{0\}$, if $x(i) \in \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$, for every i, $1 \le i \le n$.

2.5.2 Strategies that well support the payoffs of Nash equilibria

The following theorem states that the payoffs of every Nash equilibrium can be nearly well supported by a strategy profile with supports of constant size.

Theorem 2.19 *Let (x^*, y^*) be a Nash equilibrium. For every $\delta > 0$, there* *are* $\kappa(\delta)$*-uniform strategies* x, y *such that the strategy profile* (x, y) *well supports* $\big((x^*)^T R y^* - \delta, (x^*)^T C y^* - \delta\big)$*, where* $\kappa(\delta) = \lceil 2\ln(1/\delta)/\delta^2 \rceil$.

The proof of this technical result is postponed until Section 2.5.4.

Let $v, u \in [0,1]$, $\delta > 0$, and let $\mathcal{I}$ and $\mathcal{J}$ be multisets of pure strategies of size $\kappa(\delta)$. Consider the following system $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints:

$$
\begin{aligned}
x(i) &= k_i/\kappa(\delta) && \text{for all } i = 1, 2, \ldots, n;\\
y(j) &= \ell_j/\kappa(\delta) && \text{for all } j = 1, 2, \ldots, n;\\
R_{i\bullet} y &\geq v - \delta && \text{for all } i \in \mathcal{I};\\
x^T C_{\bullet j} &\geq u - \delta && \text{for all } j \in \mathcal{J};
\end{aligned}
$$

where k_i is the number of times i occurs in multiset $\mathcal{I}$, and ℓ_j is the number of times j occurs in multiset $\mathcal{J}$. Note that the system $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ of linear constraints characterizes $\kappa(\delta)$-uniform strategy profiles (x, y), such that $\mathrm{supp}(x) = \mathcal{I}$ and $\mathrm{supp}(y) = \mathcal{J}$, that well support $(v-\delta, u-\delta)$. Theorem 2.19 implies the following.

Corollary 2.20 *If* (x, y) *is a Nash equilibrium,* $v \leq x^T R y$*,* $u \leq x^T C y$*, and* $\delta > 0$*, then there are multisets* $\mathcal{I}$ *and* $\mathcal{J}$ *from* $\{1, 2, \ldots, n\}$ *of size* $\kappa(\delta)$*, such that* $\mathrm{WS}(v, u, \mathcal{I}, \mathcal{J}, \delta)$ *has a solution and it well supports* $(v - \delta, u - \delta)$.

2.5.3 The algorithm for symmetric games

Propositions 1.14 and 1.16 suggest that in order to identify an additive 1/2-well-supported Nash equilibrium it suffices to find either a strategy profile that prevents exceeding $(1/2, 1/2)$ or one that well supports $(1/2, 1/2)$. Moreover, verifying existence and identifying such strategy profiles can be done efficiently by solving the linear program $\mathrm{PE}(1/2, 1/2)$, and by solving linear programs WS$(1/2+\delta, 1/2+\delta, \mathcal{I}, \mathcal{J}, \delta)$ for all multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, respectively.

For arbitrary bimatrix games the above scheme may fail if none of these systems of linear constraints has a solution. Note, however, that—by Proposition 1.13 and Corollary 2.20—it would indeed succeed if we could guarantee that the game had a Nash equilibrium with both payoffs at most 1/2, or with both payoffs at least $(1/2+\delta)$. Symmetric bimatrix games nearly satisfy this requirement thanks to existence of symmetric Nash equilibria in every symmetric game [45].

If (x^*, x^*) is a symmetric Nash equilibrium in a symmetric bimatrix game (R, R^T) then—trivially—either $(x^*)^T Rx^* \leq 1/2$ or $(x^*)^T Rx^* > 1/2$. In the former case, by Proposition 1.13 the linear program PE$(1/2, 1/2)$ has a solution, and by Proposition 1.14 it is an additive 1/2-well-supported Nash equilibrium. In the latter case, by Corollary 2.20 there are multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies of size $\kappa(\delta)$, such that WS$(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution (x, y) and it well supports $(1/2-\delta, 1/2-\delta)$. It then follows by Proposition 1.16 that (x, y) is an additive $(1/2+\delta)$-well-supported Nash equilibrium.

Algorithm 1 *Let (R, R^T) be a symmetric game and let $\delta > 0$.*

1. *If* PE$(1/2, 1/2)$ *has a solution x then return (x, x).*
2. *Otherwise, that is if PE$(1/2, 1/2)$ does not have a solution:*
   (a) *Using exhaustive search, find multisets $\mathcal{I}$ and $\mathcal{J}$ of pure strategies, both of size $\kappa(\delta)$, such that WS$(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ has a solution.*
   (b) *Return a solution (x, y) of WS$(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$.* □

In order to find appropriate $\mathcal{I}$ and $\mathcal{J}$ in step 2(a), an exhaustive enumeration of all pairs of multisets $\mathcal{I}$ and $\mathcal{J}$ of size $\kappa(\delta)$ is done, and for each such pair the system of linear constraints $WS(1/2, 1/2, \mathcal{I}, \mathcal{J}, \delta)$ is solved. Note that the number of $\kappa(\delta)$-element multisets from an n-element set is

$$\binom{n+\kappa(\delta)-1}{\kappa(\delta)} = n^{\mathcal{O}(\kappa(\delta))} = n^{\mathcal{O}(\ln(1/\delta)/\delta^2)}.$$

Therefore, step 2. of the algorithm requires solving $n^{\mathcal{O}(\ln(1/\delta)/\delta^2)}$ linear programs and hence the algorithm runs in time $N^{\mathcal{O}(\ln(1/\delta)/\delta^2)}$.

Theorem 2.21 *For every $\delta > 0$, Algorithm 1 runs in time $N^{\mathcal{O}(\ln(1/\delta)/\delta^2)}$ and it returns a strategy profile that is an additive $(1/2+\delta)$-well-supported Nash equilibrium.*

2.5.4 Proof of Theorem 2.19

We use the probabilistic method: random $\kappa(\delta)$-uniform strategies are drawn by sampling $\kappa(\delta)$ pure strategies (with replacement) from the distributions x^* and y^*, respectively, and Hoeffding's inequality is used to show that the probability of thus selecting a strategy profile that well supports $\big(v^*-\delta, u^*-\delta\big)$ is positive if $\kappa(\delta) \geq 2\ln(1/\delta)/\delta^2$, where $v^* = (x^*)^T R y^*$ and $u^* = (x^*)^T C y^*$.

Consider $2\kappa(\delta)$ mutually independent random variables I_t and J_t, $1 \leq t \leq \kappa(\delta)$, with values in $\{1, 2, \ldots, n\}$, the former with the same distribution as strategy x^* and the latter with the same distribution as strategy y^*, that is we have $\mathbb{P}\{I_t = i\} = x_i^*$ and $\mathbb{P}\big\{J_t = j\big\} = y_j^*$ for $i, j = 1, 2, \ldots, n$. Define the random distributions $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$, with values in $[0,1]^n$, by setting:

$$X_i = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [I_t = i] \qquad \text{and} \qquad Y_j = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j].$$

Note that every realization of Y is a $\kappa(\delta)$-uniform strategy that uses the pure strategy j, $1 \leq j \leq n$, with probability $K_j/\kappa(\delta)$, where $K_j = \sum_{t=1}^{\kappa(\delta)} [J_t = j]$

is the number of indices t, $1 \le t \le \kappa(\delta)$, for which $J_t = j$. A similar characterization holds for every realization of X. Observe also that $\mathrm{supp}(X) \subseteq \mathrm{supp}(x^*)$ and $\mathrm{supp}(Y) \subseteq \mathrm{supp}(y^*)$ because for all i and j, $1 \le i, j \le n$, the random variables X_i and Y_j are identically equal to 0 unless $x_i^* > 0$ and $y_j^* > 0$, respectively.

Since we want (a realization of) the random strategies X and Y to well support a certain pair of values, we now characterize $R_{i\bullet}Y$, for all $i \in \mathrm{supp}(x^*)$; the whole reasoning presented below for $R_{i\bullet}Y$ can be carried out analogously for $X^T C_{\bullet j}$, for all $j = 1, 2, \ldots, n$, and hence it is omitted.

First, observe that for all $i = 1, 2, \ldots, n$, we have:

$$R_{i\bullet}Y = \sum_{j=1}^{n} R_{ij} Y_j = \frac{1}{\kappa(\delta)} \cdot \sum_{j=1}^{n} R_{ij} \cdot \sum_{t=1}^{\kappa(\delta)} [J_t = j] = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} R_{iJ_t}.$$

Therefore, the random variable $R_{i\bullet}Y$ is equal to the arithmetic average

$$\overline{Z_i} = \frac{1}{\kappa(\delta)} \cdot \sum_{t=1}^{\kappa(\delta)} Z_{it}$$

of the independent random variables $Z_{it} = R_{iJ_t}$, $1 \le t \le \kappa(\delta)$.

For every $i \in \mathrm{supp}(x^*)$, we will apply Hoeffding's inequality to the corresponding random variable $\overline{Z_i}$. Hoeffding's inequality gives an exponential upper bound for the probability of large deviations of the arithmetic average of independent and bounded random variables from their expectation.

Lemma 2.22 (Hoeffding's inequality) *Let $Z_1, Z_2, \ldots, Z_k$ be independent random variables with $0 \le Z_t \le 1$ for every t, let $\overline{Z} = (1/k) \cdot \sum_{t=1}^{k} Z_t$, and let $\mathbb{E}\{\overline{Z}\}$ be its expectation. Then for all $\delta > 0$, we have $\mathbb{P}\{\overline{Z} - \mathbb{E}\{\overline{Z}\} \le -\delta\} \le e^{-2\delta^2 k}$.*

Before we apply Hoeffding's inequality to the random variables $\overline{Z_i}$ defined above, observe that for every $t = 1, 2, \ldots, \kappa(\delta)$, we have:

$$\mathbb{E}\{Z_{it}\} = \mathbb{E}\{R_{iJ_t}\} = \sum_{j=1}^{n} R_{ij} \cdot \mathbb{P}\{J_t = j\} = R_{i\bullet} y^*.$$

Note, however, that if $i \in \text{supp}(x^*)$ then $\mathbb{E}\{Z_{it}\} = R_{i\bullet}y^* = v^*$, because (x^*, y^*) is a Nash equilibrium, and hence every $i \in \text{supp}(x^*)$ is a best response to y^*. It follows that $\mathbb{E}\{\overline{Z_i}\} = (1/\kappa(\delta)) \cdot \sum_{t=1}^{\kappa(\delta)} \mathbb{E}\{Z_{it}\} = v^*$.

Applying Hoeffding's inequality, for every $i \in \text{supp}(x^*)$, we get:

$$\mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} \;=\; \mathbb{P}\{\overline{Z_i} - \mathbb{E}\{\overline{Z_i}\} < -\delta\} \;\leq\; e^{-2\delta^2\kappa(\delta)}. \tag{2.24}$$

It follows that if $I \subseteq \text{supp}(x^*)$ and $|I| \leq \kappa(\delta)$, then:

$$\begin{aligned}&\mathbb{P}\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\} \;\leq \\ &\leq \sum_{i\in I} \mathbb{P}\{R_{i\bullet}Y < v^* - \delta\} \;\leq\; \kappa(\delta)\cdot e^{-2\delta^2\kappa(\delta)} \;=\; 2\delta^2\ln(1/\delta) \;<\; \frac{1}{2},\end{aligned} \tag{2.25}$$

for all $\delta > 0$. The first inequality holds by the union bound, and the second follows from (2.24) and because $|I| \leq \kappa(\delta)$. The last inequality can be verified by observing that the function $f(x) = 2x^2\ln(1/x)$, for $x > 0$, achieves its maximum at $x = 1/\sqrt{e}$ and $f(1/\sqrt{e}) = 1/e < 1/2$.

In a similar way we can prove that if $J \subseteq \text{supp}(y^*)$ and $|J| \leq \kappa(\delta)$, then:

$$\mathbb{P}\{X^T C_{\bullet j} < (x^*)^T C y^* - \delta \text{ for some } j \in J\} \;<\; \frac{1}{2}, \tag{2.26}$$

for all $\delta > 0$.

We are now ready to argue that

$$\begin{aligned}&\mathbb{P}\{R_{i\bullet}Y \geq v^* - \delta \text{ for all } i \in \text{supp}(X), \\ &\qquad\qquad \text{and } X^T C_{\bullet j} \geq u^* - \delta \text{ for all } j \in \text{supp}(Y)\} > 0,\end{aligned}$$

and hence there must be realizations $x, y \in [0,1]^n$ of the random variables $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$, such that (x, y) well supports $\big(v^* - \delta, u^* - \delta\big)$. Indeed, we have:

$$
\begin{aligned}
&\mathbb{P}\big\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in \text{supp}(X),\\
&\qquad\qquad\qquad \text{or } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in \text{supp}(Y)\big\}\\
&\quad\le \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\big\{I = \text{supp}(X) \text{ and } R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I\big\}\\
&\quad + \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\big\{J = \text{supp}(Y) \text{ and } X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J\big\}\\
&= \sum_{\substack{I \subseteq \text{supp}(x^*)\\ |I| \le \kappa(\delta)}} \mathbb{P}\big\{I = \text{supp}(X)\big\} \cdot \mathbb{P}\big\{R_{i\bullet}Y < v^* - \delta \text{ for some } i \in I \;\big|\; I = \text{supp}(X)\big\}\\
&+ \sum_{\substack{J \subseteq \text{supp}(y^*)\\ |J| \le \kappa(\delta)}} \mathbb{P}\big\{J = \text{supp}(Y)\big\} \cdot \mathbb{P}\big\{X^T C_{\bullet j} < u^* - \delta \text{ for some } j \in J \;\big|\; J = \text{supp}(Y)\big\}\\
&\; < \sum_{I \subseteq \text{supp}(x^*)} \mathbb{P}\big\{I = \text{supp}(X)\big\} \cdot \frac{1}{2} + \sum_{J \subseteq \text{supp}(y^*)} \mathbb{P}\big\{J = \text{supp}(Y)\big\} \cdot \frac{1}{2} \;=\; 1,
\end{aligned}
$$

where the first inequality follows from the union bound, and from $\text{supp}(X) \subseteq \text{supp}(x^*)$ and $\text{supp}(Y) \subseteq \text{supp}(y^*)$; the equality holds because $|\text{supp}(X)| \le \kappa(\delta)$ and $|\text{supp}(Y)| \le \kappa(\delta)$ by the definitions of X and Y; and the latter (strict) inequality follows from (2.25) and (2.26).

2.6 Additive ε-Nash equilibria in *symmetric* multi-player games

In this section, we note that we can obtain a slightly better approximation bound for *additive* ε-Nash equilibria for symmetric multi-player games than the general case using the recursive algorithm described earlier in [7, 9, 35]. The only observation is that if we have a symmetric m-player game, if we fix any strategy of any player, then the subgame that is created is also a symmetric game of $m-1$ players. And since for bimatrix games we can construct an additive $(\frac{1}{3} + \delta)$-Nash equilibrium, for any $\delta > 0$, which is a better bound than the state of art for additive ε-Nash equilibria of 0.3393 [51], using the recursive method presented in [7, 9, 35], we obtain additive $(\frac{3}{5} + \delta)$-Nash equilibria for three players for any $\delta > 0$, additive $(\frac{5}{7} + \delta)$-Nash equilibria for four players for any $\delta > 0$, and so on. In particular, by applying Lemma 9 from [7], we obtain that for any fixed m and any $\delta > 0$, one can obtain an additive $(1 - \frac{2}{2m-1} + \delta)$-Nash equilibrium in symmetric game with m players. These results are also presented in [14].

2.7 Additive ε-well-supported Nash equilibria in multi-player games

In this section, we will combine the approaches developed earlier in [42, 35, 5] for additive ε-Nash equilibria and the work of Kontogiannis and Spirakis [38] for additive ε-well-supported Nash equilibria, to prove that for any n-strategies m-player game and for any $\varepsilon > 0$, there is an ε-well-supported Nash equilibrium with support of size $O((\ln m + \ln n + \ln(1/\varepsilon)/\varepsilon^2)))$, assuming, wlog, that $mn > 8$. Results of this section are also presented in [14].

Fix an m-player game. Let $x = (x_i, x_{-i})$ be a Nash equilibrium of the game. Any player i can create an empirical distribution s_i from x_i by taking $k \geq \lceil 8(2(\ln m + \ln n) + \ln(1/\varepsilon))/\varepsilon^2 \rceil$ random samples independently according to the distribution x_i and then create a multiset A_i of pure strategies in $\{1, \ldots, n\}$. The probability of a pure strategy in the distribution s_i is the number of the appearances in the multiset A_i divided by the total number of samples k. By Lemma 3.4 from [5], for any strategy ℓ and any player i, we obtain the following:

$$\mathbf{Pr}\Big[|u_i(e_\ell, s_{-i}) - u_i(e_\ell, x_{-i})| \geq \varepsilon/2\Big] \leq \frac{8e^{-\frac{\varepsilon^2 k}{8}}}{\varepsilon} \leq \frac{8}{(mn)^2}.$$

We apply the union bound, to obtain that for the strategy profile s, we have

$$1-\sum_{i=1}^{m}\sum_{l=1}^{n}\mathbf{Pr}\Big[|u_i(e_\ell,s_{-i})-u_i(e_\ell,x_{-i})|\geq\varepsilon/2\Big]$$
$$\geq 1-\frac{8mn}{(mn)^2}=1-\frac{8}{mn}>0.$$

We apply this bound to prove the following theorem.

Theorem 2.23 *Consider an n-strategies m-player game. Let s be an empirical strategy profile, created by sampling as described above with number of samples equal to $k\geq\lceil 8(2(\ln m+\ln n)+\ln(1/\varepsilon))/\varepsilon^2\rceil$. Then s is an additive ε-well-supported Nash equilibrium, with high probability, with support of size $O(\ln m+\ln n+\ln(1/\varepsilon))/\varepsilon^2$.*

Proof. By the definition of the Nash equilibrium, for any player i, for any $\ell\in \mathit{supp}(s_i)$, $\ell'\in\{1,2,\ldots,n\}$, since $\mathit{supp}(s_i)\subseteq \mathit{supp}(x_i)$, we have that any strategy $\ell\in \mathit{supp}(s_i)$ is a best-response strategy to x_{-i}, and therefore,

$$u_i(e_\ell,x_{-i})\geq u_i(e_{\ell'},x_{-i}). \tag{2.27}$$

If we have two strategies ℓ,ℓ' such that $\ell\in \mathit{supp}(s_i)$ and $\ell'\notin \mathit{supp}(s_i)$, then

$$\begin{aligned}u_i(e_\ell,s_{-i})&\geq u_i(e_\ell,x_{-i})-\varepsilon/2\\ &\geq u_i(e_{\ell'},x_{-i})-\varepsilon/2\geq u_i(e_{\ell'},s_{-i})-\varepsilon,\end{aligned}$$

where the first and the third inequality holds by the fact that $|u_i(e_\ell,s_{-i})-u_i(e_\ell,x_{-i})|\leq\varepsilon/2$ and $|u_i(e_{\ell'},s_{-i})-u_i(e_{\ell'},x_{-i})|\leq\varepsilon/2$, respectively, and the second inequality holds by (2.27). This implies that $u_i(e_\ell,s_{-i})+\varepsilon\geq u_i(e_{\ell'},s_{-i})$.

Let us now consider two strategies $\ell, \ell' \in supp(s_i)$. Then, similarly, we have

$$\begin{aligned} u_i(e_\ell, s_{-i}) &\leq \varepsilon/2 + u_i(e_\ell, x_{-i}) = \varepsilon/2 + u_i(e_{\ell'}, x_{-i}) \\ &\leq u_i(e_{\ell'}, s_{-i}) + \varepsilon, \end{aligned}$$

where the inequalities hold because $|u_i(e_\ell, s_{-i}) - u_i(e_\ell, x_{-i})| \leq \varepsilon/2$ and $|u_i(e_{\ell'}, s_{-i}) - u_i(e_{\ell'}, x_{-i})| \leq \varepsilon/2$ and the equality holds since ℓ, ℓ' are best-response strategies to x_{-i}, since they belong to $supp(s_i) \subseteq supp(x_i)$. □

2.8 Additive approximate Nash equilibria in *random* multi-player games

In this section we will generalize the approach of [47] who gave a simple proof that random bimatrix games are easy to solve (see also [6]) to multi-player normal-form games. These results are also presented in [14].

We consider an *n-strategies m-player random normal-form game*. In a random game, every entry in all of the payoff matrices is drawn independently according to some probability distribution over the interval $[0, 1]$. We will require that for any player i the probability distributions, with which the entries are taken, have the same mean value μ_i. That is, for any player i, for any strategy profile s, $\mathbf{E}[u_i(s)] = \mu_i$.

A *fully uniform distribution* is a mixed strategy in which player plays every pure strategy with probability $\frac{1}{n}$. We will prove that, for any $\varepsilon > 0$, with high probability the fully uniform distribution is an additive ε-Nash equilibrium and an additive ε-well-supported Nash equilibrium.

Our analysis will rely on the classic Hoeffding's inequality.

Lemma 2.24 *[36] Let $Y_1, \dots, Y_\ell$, $Z_1, \dots, Z_r$ be independent random variables in the interval $[a, b]$. Let $\bar{Y} = \frac{Y_1+\cdots+Y_\ell}{\ell}$ and $\bar{Z} = \frac{Z_1+\cdots+Z_r}{r}$. Then for* *any* $t > 0$,

$$\mathbf{Pr}\Big[(\bar{Y} - \bar{Z}) - (E[\bar{Y}] - E[\bar{Z}]) \geq t\Big] \leq e^{-2t^2/((\frac{1}{\ell}+\frac{1}{r})(b-a)^2)}.$$

2.8.1 Additive ε-Nash equilibria in *random* multi-player games

First, we do the analysis for additive ε-Nash equilibria. The idea is to apply Hoeffding's inequality to show that for any player i, if she plays according to the fully uniform distribution, then if she changes her strategy then her utility will not change by more than ε. (We do not need to assume that ε is constant and the proof works for any $\varepsilon \geq \sqrt{\frac{\ln(nm)}{n^{m-1}}}$.)

Theorem 2.25 *Consider an* n*-strategies* m*-player random normal-form game and let* $\varepsilon > 0$*. Then with high probability, the strategy profile* $\mathbf{x} = (x, \ldots, x)$ *is an additive* ε*-Nash equilibrium, where* x *is a fully uniform distribution.*

Proof. Let us introduce some useful notation, and define

$$\mathcal{A}_i = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} u_i(e_{j_1}, e_{j_2}, \ldots, e_{j_m}),$$

$$\mathcal{Y}_{i,k} = \sum_{\substack{1 \leq j_1, \ldots, j_{i-1}, \\ j_{i+1}, \ldots, j_m \leq n}} u_i(e_{j_1}, \ldots, e_{j_{i-1}}, e_k, e_{j_{i+1}}, \ldots, e_{j_m}), \text{ and}$$

$$\mathcal{Z}_{i,k} = \mathcal{A}_i - \mathcal{Y}_{i,k}.$$

Observe that for every player i, we have

$$u_i(\mathbf{x}) = u_i(x, \mathbf{x}_{-i}) = \frac{1}{n^m}\mathcal{A}_i,$$

and similarly, for every player i and every strategy k,

$$u_i(e_k, \mathbf{x}_{-i}) = \frac{1}{n^{m-1}}\mathcal{Y}_{i,k}.$$

Further, after rearranging, we obtain

$$\begin{aligned} u_i(e_k,\mathbf{x}_{-i}) - u_i(x,\mathbf{x}_{-i}) &= \frac{n-1}{n^m}\mathcal{Y}_{i,k} - \frac{1}{n^m}\mathcal{Z}_{i,k} \\ &= \frac{n-1}{n} \cdot \left(\frac{1}{n^{m-1}}\mathcal{Y}_{i,k} - \frac{1}{(n-1)n^{m-1}}\mathcal{Z}_{i,k}\right). \end{aligned} \tag{2.28}$$

Now, we will bound the probability that for a given player i and strategy k, we have $u_i(e_k,\mathbf{x}_{-i}) - u_i(x,\mathbf{x}_{-i}) \geq \varepsilon$. We first want to provide the platform for the use of Hoeffding's inequality (Lemma 2.24). Notice that all entries $u_i(e_{j_1}, e_{j_2}, \ldots, e_{j_m})$ are chosen independently at random, and so, for fixed i and k, we have n^{m-1} random variables in $[0,1]$ that define $\mathcal{Y}_{i,k}$ and $(n-1)n^{m-1}$ random variables in $[0,1]$ that define $\mathcal{Z}_{i,k}$, and all these random variables are independent. The random variables defining $\mathcal{Y}_{i,k}$ will correspond to Y_js and random variables defining $\mathcal{Z}_{i,k}$ will correspond to Z_js in Lemma 2.24, and so, we will have $\bar{Y} = \frac{1}{n^{m-1}} \cdot \mathcal{Y}_{i,k}$ and $\bar{Z} = \frac{1}{(n-1)n^{m-1}} \cdot \mathcal{Z}_{i,k}$.

Further, let us notice that

$$\begin{aligned} \mathbf{E}[\bar{Y}] &= \mathbf{E}\big[\frac{1}{n^{(m-1)}}\mathcal{Y}_{i,k}\big] = \frac{n^{(m-1)}\mu_1}{n^{(m-1)}} = \mu_1, \\ \mathbf{E}[\bar{Z}] &= \mathbf{E}\big[\frac{1}{(n-1)n^{(m-1)}}\mathcal{Z}_{i,k}\big] = \frac{(n-1)n^{(m-1)}\mu_1}{(n-1)n^{(m-1)}} = \mu_1. \end{aligned}$$

Therefore, since $(\bar{Y} - \bar{Z}) - (\mathbf{E}[\bar{Y}] - \mathbf{E}[\bar{Z}]) = \bar{Y} - \bar{Z}$, identity (2.28) gives us the following:

$$\begin{aligned} &\mathbf{Pr}\Big[u_i(e_k,\mathbf{x}_{-i}) - u_i(x,\mathbf{x}_{-i}) \geq \varepsilon\Big] \\ &\qquad = \mathbf{Pr}\big[\frac{1}{n^{m-1}}\mathcal{Y}_{i,k} - \frac{1}{(n-1)n^{m-1}}\mathcal{Z}_{i,k} \geq \frac{n}{n-1}\varepsilon\big] \\ &\qquad = \mathbf{Pr}\Big[(\bar{Y} - \bar{Z}) - (\mathbf{E}[\bar{Y}] - \mathbf{E}[\bar{Z}]) \geq \frac{n}{n-1}\varepsilon\Big]. \end{aligned}$$

Now Hoeffding's inequality (Lemma 2.24) gives that for any fixed player i and any fixed strategy k,

$$\mathbf{Pr}\Big[u_i(e_k,\mathbf{x}_{-i}) - u_i(x,\mathbf{x}_{-i}) \geq \varepsilon\Big] \leq e^{\frac{-2n^m\varepsilon^2}{n-1}} \leq e^{-2n^{m-1}\varepsilon^2}.$$

Therefore, by the union bound,

$$\mathbf{Pr}\Big[\exists_i \exists_k u_i(e_k, \mathbf{x}_{-i}) - u_i(x, \mathbf{x}_{-i}) \geq \varepsilon\Big] \leq nm\ e^{-2n^{m-1}\varepsilon^2}.$$

Thus for any $0 < \delta < 1$, if we set $\varepsilon = \sqrt{\frac{\ln(nm/\delta)}{2n^{m-1}}}$, then

$$\mathbf{Pr}\Big[\mathbf{x} \text{ is an additive } \varepsilon\text{-Nash equilibrium}\Big] \geq 1 - \delta.$$

Hence, if we choose $\delta = \frac{1}{nm}$, then with high probability, the strategy profile $\mathbf{x}$ is an additive ε-Nash equilibrium for $\varepsilon \geq \sqrt{\frac{\ln(nm)}{n^{m-1}}}$. □

2.8.2 Additive ε-well-supported Nash equilibria in *random* multi-player games

The analysis from Theorem 2.25 can be easily extended to approximate *well-supported* Nash equilibria in random multi-player games.

Theorem 2.26 *Consider an n-strategies m-player random normal-form game and let $\varepsilon > 0$. Then with high probability, the strategy profile $\mathbf{x} = (x, \dots, x)$ is an additive ε-well-supported Nash equilibrium, where x is a fully uniform distribution.*

As in Theorem 2.25, ε does not need to be constant and it may be as small as $\varepsilon = \sqrt{\frac{3\ln(nm)}{n^{m-1}}}$.

Proof. The proof mimics the analysis from Theorem 2.25, but this time we focus on well-supported Nash equilibria.

We use the notation from the proof of Theorem 2.25. With this, for a fixed player i and fixed strategies ℓ, k, we have,

$$u_i(e_\ell, \mathbf{x}_{-i}) - u_i(e_k, \mathbf{x}_{-i}) = \frac{1}{n^{m-1}}\mathcal{Y}_{i,\ell} - \frac{1}{n^{m-1}}\mathcal{Y}_{i,k}.$$

We will want to use Hoeffding's inequality with $\bar{Y} = \frac{1}{n^{m-1}}\mathcal{Y}_{i,\ell}$ and $\bar{Z} = \frac{1}{n^{m-1}}\mathcal{Y}_{i,k}$, and after observing that $\mathbf{E}[\bar{Y}] = \mathbf{E}[\bar{Z}] = \mu_i$, and

$$\mathbf{Pr}\Big[u_i(e_\ell, \mathbf{x}_{-i}) - u_i(e_k, \mathbf{x}_{-i}) \geq \varepsilon\Big]$$
$$= \mathbf{Pr}\Big[(\bar{Y} - \bar{Z}) - (\mathbf{E}[\bar{Y}] - \mathbf{E}[\bar{Z}]) \geq \varepsilon\Big],$$

we can apply Hoeffding's inequality (Lemma 2.24) to obtain

$$\mathbf{Pr}\Big[u_i(e_\ell, \mathbf{x}_{-i}) - u_i(e_k, \mathbf{x}_{-i}) \geq \varepsilon\Big] \leq e^{-\varepsilon^2 n^{m-1}}.$$

Therefore, by the union bound we obtain the following:

$$\mathbf{Pr}\Big[\exists_i \exists_\ell \exists_k u_i(e_\ell, \mathbf{x}_{-i}) - u_i(e_k, \mathbf{x}_{-i}) \geq \varepsilon\Big] \leq mn^2 \cdot e^{-\varepsilon^2 n^{m-1}}.$$

So, for any $0 < \delta < 1$, if we set $\varepsilon = \sqrt{\frac{\ln(n^2 m/\delta)}{n^{m-1}}}$, then

$$\mathbf{Pr}\Big[\mathbf{x} \text{ is an additive } \varepsilon\text{-well-supported Nash equilibrium}\Big] \geq 1 - \delta.$$

Therefore, if we choose $\delta = \frac{1}{nm}$, then with high probability, the strategy profile $\mathbf{x}$ is an additive ε-well-supported Nash equilibrium for $\varepsilon \geq \sqrt{\frac{3\ln(nm)}{n^{m-1}}}$.

□

Chapter 3

Relative ε-Nash equilibria

While the additive approximation of Nash equilibria has been studied much in the past, significant less attention has been paid in the relative approximate Nash equilibria. A relative ε-Nash equilibrium is a strategy profile in which the expected payoff of any player is at least $(1 - \varepsilon)$ times the best-response expected payoff. In this chapter, we will present a polynomial-time algorithm for computing relative 1/2-Nash equilibria in bimatrix games and we will generalize it in the case of multi-player games. Results of this chapter also appeared in [14].

3.1 Finding relative $\frac{1}{2}$-Nash equilibria in bimatrix games

We will give a simple algorithm to find a relative $\frac{1}{2}$-Nash equilibrium in bimatrix games. We consider the strategy profile similar to that already studied by Feder et al. [28]. Let R_{ij} be the maximum entry in array R. Let C_{ic} be the maximum entry in row i in array C; in other words, strategy c is the best-response strategy of the column player to the strategy i of the row player.

Claim 3.1 *The strategy profile* $(i, \frac{1}{2}j + \frac{1}{2}c)$ *is a relative* $\frac{1}{2}$*-Nash equilibrium.*

Proof. Let (x, y) be the strategy profile $(i, \frac{1}{2}j + \frac{1}{2}c)$. We will show that $x^T Ry \geq \frac{1}{2}(x')^T Ry$ for every mixed strategy x' and $x^T Cy \geq \frac{1}{2}x^T Cy'$ for every mixed strategy y'.

Let us first consider the row player; her expected payoff is equal to $x^T Ry = \frac{1}{2}R_{ij} + \frac{1}{2}R_{ic} \geq \frac{1}{2}R_{ij}$. Since the maximum possible payoff for the row player is R_{ij}, we have that for every mixed strategy x' of the row player, $\frac{1}{2}(x')^T Ry \leq \frac{1}{2}R_{ij} \leq \frac{1}{2}R_{ij} + \frac{1}{2}R_{ic} = x^T Ry$.

Next we consider the column player; her expected payoff is $x^T Cy = \frac{1}{2}C_{ij} + \frac{1}{2}C_{ic}$. Since c is the best-response strategy of the column player to the strategy i of the row player, we have $x^T Ce_c \geq x^T Ce_\ell$ for every $\ell = 1, \ldots, n$, and hence $x^T Ce_c \geq x^T Cy'$ for every mixed strategy y' of the column player. Therefore, for every mixed strategy y' of the column player we have $x^T Cy = \frac{1}{2}C_{ij} + \frac{1}{2}C_{ic} = \frac{1}{2}x^T Ce_c \geq \frac{1}{2}x^T Cy'$. □

3.2 Finding relative $\left(1 - \frac{1}{1+(m-1)^m}\right)$-Nash equilibria for m-player games

We generalize the simple algorithm for relative ε-Nash equilibria of two players (Section 3.1) to multiple players. We consider an m-player normal-form game with n-strategies for any player and with a constant number of players m. The size of the description of the game is $mn^m = \mathcal{O}(n^m)$.

Theorem 3.2 *For any m-player normal-form game, where m is a constant, with entries in* $[0, 1]$*, we can construct in polynomial-time a relative* $(1 - \frac{1}{1+(m-1)^m})$*-Nash equilibrium.*

Proof. For the first $m-1$ players, we find their maximum entries in their payoff matrix in time $\mathcal{O}(n^m)$, which is in polynomial-time assuming that m is constant. The maximum entry of any of the first $m-1$ players i is a pure strategy profile $s^{\langle i\rangle} = (s_1^{\langle i\rangle}, s_2^{\langle i\rangle}, \ldots, s_m^{\langle i\rangle}) \in S_1 \times S_2 \times \cdots \times S_m$, where S_i is the set of strategies of player i. For any player i, $1 \leq i \leq m$, let $K_i \subseteq S_i$ be the set of her pure strategies in the pure strategy profiles that maximize the entries of the first $m-1$ players, that is, $K_i = \{s_i^{\langle j\rangle} : 1 \leq j \leq m-1\}$. Now, we can define our relative $(1-\frac{1}{1+(m-1)^m})$-Nash equilibrium. Any of the first $m-1$ players plays with uniform probability $\frac{1}{|K_i|}$ any strategy in her set K_i. Then, player m best-response to this strategy profile is some strategy b. Let $p = \frac{1}{(m-1)+(m-1)^{-(m-1)}}$; player m plays with probability p any strategy that is in the set K_m and plays the strategy b with probability $1-|K_m|p$.

For any i, $1 \leq i \leq m-1$, the probability that $s^{\langle i\rangle}$ is chosen is at least $p \cdot \prod_{i=1}^{m-1} \frac{1}{|K_i|}$, and therefore the expected payoff of player i is at least $p \cdot \prod_{i=1}^{m-1} \frac{1}{|K_i|} \geq p \cdot (m-1)^{-(m-1)}$ times her maximum entry. Further, the expected payoff of player m is at least $1-|K_m|p \geq 1-(m-1)p$ of her best-response payoff. Therefore, the obtained strategy profile is a relative ε-Nash equilibrium for $\varepsilon = 1-\min\{p \cdot (m-1)^{-(m-1)}, 1-(m-1)p\}$. If we plug in the value of p, we obtain $p \cdot (m-1)^{-(m-1)} = 1-(m-1)p$, and that

$$(m-1)p = \frac{m-1}{m-1+(m-1)^{-(m-1)}} = \frac{(m-1)^m}{(m-1)^m+1} = 1-\frac{1}{(m-1)^m+1}.$$

Therefore,

$$\varepsilon = 1-\min\{p\cdot(m-1)^{-(m-1)}, 1-(m-1)p\} = (m-1)p = 1-\frac{1}{(m-1)^m+1}.$$

$\square$

3.3 Finding relative $\left(1 - \frac{1}{1+(m-1)^{m-1}}\right)$-Nash equilibria for symmetric m-player games

We adjust the general algorithm of section 3.2 to the case of symmetric m-player games.

Theorem 3.3 *For any symmetric m-player normal-form game, where m is a constant, with entries in $[0,1]$, we can construct in polynomial-time a relative $(1 - \frac{1}{1+(m-1)^{m-1}})$-Nash equilibrium.*

Proof. For the player 1 we find the maximum entry of her payoff matrix. In this strategy profile player m plays a pure strategy l. We keep the strategy of the player m fixed. The subgame that is created for the other $m-1$ players in the set $\{1, \ldots, m-1\}$ is a symmetric game because of the definition of the symmetric games. So, the $m-1$ players have their maximum entry in the same subgame. For any player i, $1 \le i \le m-1$, let $K_i \subseteq S_i$ be the set of her pure strategies in the pure strategy profiles that maximize the entries of the first $m-1$ players, that is, $K_i = \{s_i^{\langle j \rangle} : 1 \le j \le m-1\}$.

Now, we can define our relative $(1 - \frac{1}{1+(m-1)^{m-1}})$-Nash equilibrium. Any of the first $m-1$ players plays with uniform probability $\frac{1}{|K_i|}$ any strategy in her set K_i. Then, player m best-response to this strategy profile is some strategy b. Let player m plays with probability p the initial fixed strategy l and plays the strategy b with probability $1-p$.

For any i, $1 \le i \le m-1$, the probability that $s^{\langle i \rangle}$ is chosen is at least $p \cdot \prod_{i=1}^{m-1} \frac{1}{|K_i|}$, and therefore the expected payoff of player i is at least $p \cdot \prod_{i=1}^{m-1} \frac{1}{|K_i|} \ge p \cdot (m-1)^{-(m-1)}$ times her maximum entry. Further, the expected payoff of player m is at least $1-p$ of her best-response payoff. Therefore, the obtained strategy profile is a relative ε-Nash equilibrium for $\varepsilon = 1 - \min\{p \cdot (m-1)^{-(m-1)}, 1-p\}$. If we plug in the value of p, we obtain

$p \cdot (m-1)^{-(m-1)} = 1 - p$, and that

$$p \;=\; \frac{1}{1+(m-1)^{-(m-1)}} \;=\; \frac{(m-1)^{m-1}}{(m-1)^{m-1}+1} \;=\; 1 - \frac{1}{(m-1)^{m-1}+1}.$$

Therefore,

$$\varepsilon \;=\; 1 - \min\{p \cdot (m-1)^{-(m-1)}, 1-p\} \;=\; p \;=\; 1 - \frac{1}{(m-1)^{m-1}+1}.$$

$\square$

Part II

Optimal approximate Nash equilibria

Chapter 4

Near optimal additive ε-Nash equilibria

While the Nash theorem [45] ensures that every finite two-player game has at least one Nash equilibrium, typical games posses many equilibria and it is natural to seek those equilibria that are more desirable than others. One natural measure of the most desirable equilibria is to maximize its social welfare, that is, the sum of players' payoffs. Unlike the problem of finding a Nash equilibrium, which is known to be PPAD-complete, finding a Nash equilibrium with maximal social welfare is known to be NP-hard [30, 12, 43], and thus, it is likely to be computationally even more difficult. In fact, it is even NP-hard to approximate (to any positive ratio) the maximum social welfare obtained in an exact Nash equilibrium, even in symmetric 2-player games [12, Corollary 6]. Therefore, it is natural to ask the question of computational complexity of finding an additive ε-Nash equilibrium that approximates well the optimal social welfare. The quasi-polynomial-time algorithm by [42] not only finds an additive ε-Nash equilibrium, but also the social welfare of the equilibrium found is an ε-approximation of the social welfare in any Nash equilibrium. In other words, in quasi-polynomial-time we can find an arbitrarily good additive ε-Nash equilibrium with social welfare near to the best Nash equilibrium in bimatrix games.

4.1 Additive ε-Nash equilibria with near optimal social welfare

In this chapter, we focus on the analysis of the social welfare in additive ε-Nash equilibria in a two-player game for a fixed ε, for the regime when we know that we can find an additive ε-Nash equilibrium. Our goal is more general than that presented in earlier works, like e.g., in [4, 8, 34, 44]; it is not to compare the social welfare of an additive ε-Nash equilibrium to that of any Nash equilibrium, but rather to *compare it with the optimal social welfare*. Results of this chapter also appeared in [17].

It is known that a Nash equilibrium can be arbitrarily far from the optimal social welfare in a bimatrix game. A simple example describing this situation is a *prisoners' dilemma* game:

	C	D
C	$\left(\frac{2}{3}, \frac{2}{3}\right)$	$(0, 1)$
D	$(1, 0)$	(δ, δ)

Assuming that $\delta \in \left(0, \frac{2}{3}\right)$, the optimal social welfare is achieved by the strategy profile (C, C) with total payoff of $\frac{4}{3}$, but the unique Nash equilibrium is the strategy profile (D, D) with total payoff of 2δ. Thus, by taking δ arbitrarily small, we can make the social welfare of a Nash equilibrium arbitrarily far from the optimal social welfare of a game.

The central question studied in this chapter is if we allow the players up to ε loss to deviate from the best-response strategy, whether we can find a stable strategy profile (an additive ε-Nash equilibrium) that guarantees the players a value close to the social optimum?

We note that, to the best of our knowledge, the known polynomial-time algorithms to construct an additive ε-Nash equilibrium for a constant $\varepsilon > 0$, do not guarantee any welfare for the additive ε-Nash equilibrium and they return an additive ε-Nash equilibrium strategy profile which can be arbitrarily far from the optimal social welfare (see, e.g., [7, 20, 22, 23, 37, 51] for more details).

4.2 New contributions

In this chapter, we provide several results showing that for every bimatrix game, for every $\varepsilon > 0$, there is always an additive ε-Nash equilibrium with near optimal social welfare, at least a constant fraction the optimal social welfare. Our analysis shows that by considering an appropriate mixture of the optimal strategies and exact or additive ε-Nash equilibria, one can find the desired additive ε-Nash equilibrium with near optimal social welfare.

We begin with the case when $\varepsilon \geq \frac{1}{2}$, the case for which it is known that there is always an additive ε-Nash equilibrium with constant size support (cf. [23]). We show that in that case we can find an additive ε-Nash equilibrium with constant size support whose social welfare is at least $2\sqrt{\varepsilon} - \varepsilon \geq 0.914$ times the optimal social welfare. Furthermore, we demonstrate that our bound for the social welfare is *tight*.

Theorem 4.1 *For every $\varepsilon \geq \frac{1}{2}$, we can construct in polynomial-time an additive ε-Nash equilibrium (and with constant size support) whose social value is at least $2\sqrt{\varepsilon} - \varepsilon$ times the optimal social welfare. Furthermore, there is a bimatrix game for which for every $\varepsilon \geq \frac{1}{2}$, every additive ε-Nash* *equilibrium has social welfare no more than* $2\sqrt{\varepsilon} - \varepsilon$ *times the optimal social welfare.*

In particular, we can construct in polynomial-time an additive $\frac{1}{2}$*-Nash equilibrium whose social welfare is at least* $\frac{2\sqrt{2}-1}{2} \approx 0.914$ *times the optimal social welfare.*

As a byproduct of our approach, we also obtain a stronger result for the class of win-lose bimatrix games and show that for any $\varepsilon \in [\frac{1}{2}, 1]$, for any win-lose bimatrix game with values in $\{0, 1\}$, we can find in polynomial-time an additive ε-Nash equilibrium with optimal social welfare (Theorem 4.5).

The case $\varepsilon < \frac{1}{2}$ is more challenging and while we do not have a tight bound for the social welfare in this case, we can still construct an additive ε-Nash equilibrium with social welfare that is at least κ_ε times the optimum, for some positive constant κ_ε. One challenge in the case $\varepsilon < \frac{1}{2}$ stems from the fact that there are bimatrix games with no additive/relative ε-Nash equilibrium with constant support (cf. [28]), which requires us to use a different approach than that in Theorem 4.1 to deal with this case. Using as a starting point additive ε^*-Nash equilibria with arbitrary social welfare and $\varepsilon^* < \varepsilon$, we modify them to obtain an additive ε-Nash equilibrium with high social welfare to get the following.

Theorem 4.2 *For every fixed positive* $\varepsilon < \frac{1}{2}$ *there is a positive constant* $\kappa_\varepsilon = (1 - \sqrt{1-\varepsilon})^2$*, such that every bimatrix game has an additive* ε*-Nash equilibrium with social welfare at least* κ_ε *times the optimal social welfare.*

Our construction is algorithmic and gives the following.

Theorem 4.3 *Let* ε^* *be such that there is a polynomial-time algorithm for finding an additive* ε^**-Nash equilibrium of a bimatrix game. Then for every fixed positive* $\varepsilon > \varepsilon^*$*, there is a positive constant* $\zeta_{\varepsilon,\varepsilon^*} = (1 - \sqrt{\frac{1-\varepsilon}{1-\varepsilon^*}})^2$*, such* *that for every bimatrix game one can find in polynomial-time an additive ε-Nash equilibrium with social welfare at least $\zeta_{\varepsilon,\varepsilon^*}$ times the optimal social welfare.*

We also obtain further algorithmic results improving the bounds for the social welfare above in several special cases for $\varepsilon < \frac{1}{2}$. For example, in the case when the optimal social welfare is at least $\frac{2-3\varepsilon}{1-\varepsilon}$, then in Theorem 4.8 we design a polynomial-time algorithm that finds an additive ε-Nash equilibrium with constant support size and with social welfare at least $(1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}) \geq 0.5$ times the optimum social welfare. For this case we will prove that if the optimum social welfare is less than $\frac{2-3\varepsilon}{1-\varepsilon}$, we need logarithmic support in order to create an additive ε-Nash equilibrium.

We will prove Theorem 4.1 in Section 4.4 and Theorems 4.2 and 4.3 in Section 4.8.

4.3 Preliminaries

The *social welfare* is the total payoff of both players, i.e., it is $\mathrm{sw} = x^T Ry + x^T Cy = x^T(R+C)y$. Throughout the chapter, we let (i, j) to denote the pure strategy profile that maximizes the sum of the payoffs of the two players (utilitarian objective). We define $\mathtt{opt}$ to be the *optimal social welfare*, that is,

$$\forall x, y \in [0,1]^n, \mathtt{opt} = R_{ij} + C_{ij} \geq x^T(R+C)y. \tag{4.1}$$

(Note that i and j can be trivially found in $\mathcal{O}(n^2)$ time.)

We define the pure strategy r of the row player as the best-response strategy of the row player to the strategy j of the column player and the pure strategy c of the column player as the best-response strategy of the column player to the strategy i of the row player. The optimality of the profile (i, j) yields:

$$\mathtt{opt} = R_{ij} + C_{ij} \geq R_{ic} + C_{ic}, \tag{4.2}$$

$$\mathtt{opt} = R_{ij} + C_{ij} \geq R_{rj} + C_{rj}. \tag{4.3}$$

The central goal of this chapter is for a fixed $\varepsilon \in [0, 1]$, to find an additive ε-Nash equilibrium strategy profile (x^*, y^*) whose social welfare sw is as close to $\mathtt{opt}$ as possible.

In our analysis, we will consider several cases depending on the values of $R_{rj} - R_{ij}$ and $C_{ic} - C_{ij}$:

- $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$,
- $R_{rj} - R_{ij} \geq \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$ (and the symmetric case $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} \geq \varepsilon$),
- $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$ (and the symmetric case $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$).

We will use the fact that in the first case, when $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, the strategy profile (i, j) (which can be found in polynomial-time) is an additive ε-Nash equilibrium, and since it has the optimal social welfare, in this case we can find an optimal solution by choosing strategy (i, j). Thus, our main task will be to find a good algorithm to construct an additive ε-Nash equilibrium in the other cases.

In our analysis, we will separately consider two regimes: one when $\varepsilon \geq \frac{1}{2}$ and one when $\varepsilon < \frac{1}{2}$.

4.4 Approximation with $\varepsilon \geq \frac{1}{2}$

We begin with the scenario when $\varepsilon \geq \frac{1}{2}$, proving Theorem 4.1. We will show in Section 4.5 that if $\varepsilon \geq \frac{1}{2}$, then one can find an additive ε-Nash equilibrium with constant size support that has an *almost* optimal social welfare, at least $2\sqrt{\varepsilon} - \varepsilon \geq 0.914$ times the optimal social welfare. We will also prove that our bound is tight for any $\varepsilon \geq \frac{1}{2}$, by showing in Section 4.6 explicit bimatrix games for which every additive ε-Nash equilibrium has social welfare no more than $2\sqrt{\varepsilon} - \varepsilon$ times the optimal social welfare.

Let us recall that (i, j) is the pure strategy profile that maximizes the sum of the payoffs of the two players, and hence $\mathfrak{opt} = R_{ij} + C_{ij}$ (cf. (4.1)). Let us recall that r is the pure strategy of the row player that is the best-response strategy of the row player to the strategy j of the column player and that c is the pure strategy of the column player that is the best-response strategy of the column player to the strategy i of the row player. We will now consider several cases depending on the values of $R_{rj} - R_{ij}$ and $C_{ic} - C_{ij}$.

Let us first note that it is impossible to have $R_{rj} - R_{ij} \geq \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, or to have $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} \geq \varepsilon$ (since these cases are symmetric, we will focus only on the first one). To show that we cannot have $R_{rj} - R_{ij} \geq \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, we first observe that these inequalities yield:

$$R_{ij} \leq R_{rj} - \varepsilon \leq 1 - \varepsilon \text{ and } C_{ij} < C_{ic} - \varepsilon < 1 - \varepsilon. \tag{4.4}$$

Next, $R_{rj} - R_{ij} \geq \varepsilon$ together with (4.3) yield $R_{ij} + C_{ij} \geq R_{rj} + C_{rj} \geq R_{ij} + \varepsilon + C_{rj}$, which implies $C_{ij} \geq \varepsilon$. Similarly, $C_{ic} - C_{ij} > \varepsilon$ and (4.2) give $R_{ij} + C_{ij} \geq R_{ic} + C_{ic} > R_{ic} + C_{ij} + \varepsilon$, and hence $R_{ij} > \varepsilon$. Now, however, we observe that with the assumption $\varepsilon \geq \frac{1}{2}$, the inequalities above form a contradiction, and therefore this case cannot happen.

Since we cannot have either of the cases $R_{rj} - R_{ij} \geq \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, or $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} \geq \varepsilon$, we only have to consider one of the following three scenarios: (1) $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, (2) $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, (3) $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$.

We will now consider these cases, depending on the values of $R_{rj} - R_{ij}$ and $C_{ic} - C_{ij}$:

(1) If $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, then we know that the strategy profile (i, j) is an additive ε-Nash equilibrium with the optimal social welfare.

(2) If $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, then we note that

$$C_{ic} > C_{ij} + \varepsilon \geq \max\{C_{ij}, \varepsilon\}, \tag{4.5}$$

and that (4.2) yields

$$R_{ij} - R_{ic} \geq C_{ic} - C_{ij} > \varepsilon. \tag{4.6}$$

Next, we prove a key lemma describing an additive ε-Nash equilibrium in our setting.

Lemma 4.4 *Let $\varepsilon \in [\frac{1}{2}, 1]$, $R_{rj} - R_{ij} < \varepsilon$, and $C_{ic} - C_{ij} > \varepsilon$. Let $\mathfrak{p} = \frac{\varepsilon}{C_{ic} - C_{ij}}$. The strategy profile $(i, \mathfrak{p}j + (1 - \mathfrak{p})c)$, where $\mathfrak{p}$ is the probability for the column player to play strategy j and $(1 - \mathfrak{p})$ is the probability of playing strategy c respectively, is an additive ε-Nash equilibrium.*

Proof. Let us first notice that $\mathfrak{p}$ is well defined with $0 < \mathfrak{p} \leq 1$ since $0 < \varepsilon < C_{ic} - C_{ij}$.

Let b be the best-response strategy of the row player to the strategy $\mathfrak{p}j + (1 - \mathfrak{p})c$ of the column player. If the row player plays strategy i, her incentive to deviate is:

$$\begin{aligned} \mathfrak{p}R_{bj} + (1-\mathfrak{p})R_{bc} - \mathfrak{p}R_{ij} - (1-\mathfrak{p})R_{ic} &\leq \mathfrak{p}R_{rj} + (1-\mathfrak{p}) - \mathfrak{p}R_{ij} - (1-\mathfrak{p})R_{ic} \\ &\leq \mathfrak{p} + (1 - \mathfrak{p}) - \mathfrak{p}(R_{ij} - R_{ic}) = 1 - \frac{\varepsilon \cdot (R_{ij} - R_{ic})}{C_{ic} - C_{ij}} \leq 1 - \varepsilon \leq \varepsilon. \end{aligned}$$

The first inequality follows from $R_{bj} \leq R_{rj}$ and $R_{bc} \leq 1$, the second one because of the fact that $R_{rj} \leq 1$ and $R_{ic} \geq 0$, the third one because $R_{ij} + C_{ij} \geq R_{ic} + C_{ic}$, and the final one follows from the fact that $\varepsilon \geq \frac{1}{2}$.

On the other hand, the incentive to deviate for the column player when the row player plays i is $C_{ic} - \mathfrak{p}C_{ij} - (1 - \mathfrak{p})C_{ic} = \varepsilon$. Hence the strategy profile $(i, \mathfrak{p}j + (1 - \mathfrak{p})c)$ is an additive ε-Nash equilibrium. $\square$

(3) $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$ is symmetric to case (2).

4.5 Upper bound in Theorem 4.1

We now prove that ε-Nash (R, C, ε) presented below, returns an additive ε-Nash equilibrium with social welfare at least $(2\sqrt{\varepsilon} - \varepsilon) \cdot \mathsf{opt}$. By the arguments above, we only have to consider the following scenarios: (1) $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, (2) $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, (3) $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$.

ε-Nash (R, C, ε)

- Find i, j such that $R_{ij} + C_{ij}$ is maximized.
- Find r, c such that R_{rj} is maximized and C_{ic} is maximized.
- If $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, then return strategy profile (i, j).
- If $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, then set $\mathfrak{p} = \frac{\varepsilon}{C_{ic} - C_{ij}}$ and return strategy profile $(i, \mathfrak{p}j + (1 - \mathfrak{p})c)$.
- If $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$, then set $\mathfrak{p} = \frac{\varepsilon}{R_{rj} - R_{ij}}$ and return strategy profile $(\mathfrak{p}i + (1 - \mathfrak{p})r, j)$.

Let us recall that if $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, then the strategy (i, j) is an additive ε-Nash equilibrium with social welfare $\mathtt{opt}$, and therefore the algorithm will return an optimum solution that is an additive ε-Nash equilibrium. Therefore, we only have to consider scenarios (2) and (3). Since these scenarios are symmetric, we focus only on scenario (2), when $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$: we prove that the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$ with $\mathfrak{p} = \frac{\varepsilon}{C_{ic}-C_{ij}}$ has social welfare at least $(2\sqrt{\varepsilon} - \varepsilon) \cdot \mathtt{opt}$.

The social welfare of our solution is $\mathrm{sw} = \mathfrak{p}(R_{ij}+C_{ij})+(1-\mathfrak{p})(R_{ic}+C_{ic})$. Let

$$\begin{aligned} \rho &= \frac{\mathtt{opt}}{\mathrm{sw}} = \frac{R_{ij} + C_{ij}}{\mathfrak{p}(R_{ij} + C_{ij}) + (1 - \mathfrak{p})(R_{ic} + C_{ic})} \\ &\leq \frac{R_{ij} + C_{ij}}{\mathfrak{p}(R_{ij} + C_{ij}) + (1 - \mathfrak{p})C_{ic}}. \end{aligned} \tag{4.7}$$

Observe that if we consider the last bound as a function of R_{ij}, we obtain a function of the form $f(x) = \frac{x+\beta}{\mathfrak{p}x+\gamma}$, with $0 \leq \mathfrak{p} \leq 1$, $\beta = C_{ij}$ and $\gamma = \mathfrak{p}C_{ij} + (1 - \mathfrak{p})C_{ic}$. Notice further that since by (4.5), we have $\mathfrak{p}C_{ij} + (1 - \mathfrak{p})C_{ic} > \mathfrak{p}C_{ij} + (1 - \mathfrak{p})C_{ij} = C_{ij} \geq 0$, we obtain $\gamma > \beta \geq 0$. Therefore, by considering the derivative $f'(x) = \frac{\gamma-\mathfrak{p}\beta}{(\mathfrak{p}x+\gamma)^2} > 0$, we observe that f is increasing in x. Thus, the right hand side of (4.7) takes the maximum value when R_{ij} is maximum, that is, is equal to 1, independently from the other variables. Hence,

$$\begin{aligned} \rho &\leq \frac{1 + C_{ij}}{\mathfrak{p} + \mathfrak{p}C_{ij} + (1 - \mathfrak{p})C_{ic}} \\ &= \frac{-C_{ij}^2 - C_{ij}(1 - C_{ic}) + C_{ic}}{-C_{ij}(C_{ic} - \varepsilon) + C_{ic}(C_{ic} - \varepsilon) + \varepsilon}. \end{aligned} \tag{4.8}$$

We note that the right hand side of (4.8) takes maximum when $C_{ic} = C_{ij} + \sqrt{\varepsilon}$, and hence when $\mathfrak{p} = \sqrt{\varepsilon}$. If we plug this in (4.8), then we obtain $\rho \leq \frac{1+C_{ij}}{2\sqrt{\varepsilon}-\varepsilon+C_{ij}}$. Next, we observe that since $\varepsilon \in [\frac{1}{2}, 1]$ we have $2\sqrt{\varepsilon} - \varepsilon \leq 1$, and hence the right hand side of is decreasing and takes the maximum at

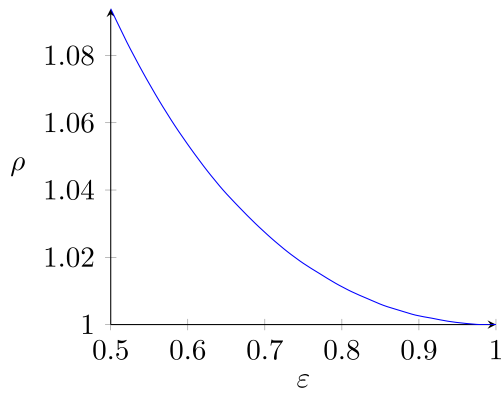

Figure 4.1: Bound for $\rho = \frac{\mathfrak{opt}}{\mathrm{sw}}$ as a function of ε, $\varepsilon \geq \frac{1}{2}$. Notice that $\rho(1) = 1$ and $\rho(\frac{1}{2}) \leq \frac{2}{2\sqrt{2}-1} \approx 1.094$.

$C_{ij} = 0$. Therefore $\rho \leq \frac{1}{2\sqrt{\varepsilon}-\varepsilon}$. This completes the proof of the first part (upper bound) of Theorem 4.1.

Figure 4.1 depicts the upper bound as a function of ε.

4.6 Lower bound in Theorem 4.1

We now show the second part of Theorem 4.1 and for every $\varepsilon \in [\frac{1}{2}, 1]$, we present a game for which the social welfare of every additive ε-Nash equilibrium is at most $(2\sqrt{\varepsilon} - \varepsilon) \cdot \mathfrak{opt}$.

Fix ε, $\frac{1}{2} \leq \varepsilon \leq 1$. Consider a bimatrix game with one strategy for the row player, strategy i, and with two strategies for the column player, strategies j and c. Set $R_{ij} = 1$, $C_{ij} = 0, R_{ic} = 0$, and $C_{ic} = \sqrt{\varepsilon}$, resulting in the following game:

	j	c
i	$(1, 0)$	$(0, \sqrt{\varepsilon})$

The optimal strategy is (i, j) with the social welfare $\mathfrak{opt} = 1$. In order to obtain an optimal additive ε-Nash equilibrium the column player needs to randomize between her strategies, playing strategy j with probability $\mathfrak{p}$ and strategy c with probability $(1-\mathfrak{p})$. Then, the strategy profile $(i, \mathfrak{p}j+(1-\mathfrak{p})c)$ is an additive ε-Nash equilibrium if and only if $\sqrt{\varepsilon} \leq (1-\mathfrak{p})\sqrt{\varepsilon}+\varepsilon$. This is equivalent to $\mathfrak{p} \leq \sqrt{\varepsilon}$. Conditioned on this, we bound the social welfare of any additive ε-Nash equilibrium for this game. For any $0 \leq \mathfrak{p} \leq \sqrt{\varepsilon}$, if we denote the social welfare of an additive ε-Nash equilibrium with fixed $\mathfrak{p}$ by $\mathrm{sw}_{\mathfrak{p}}$, then we obtain, $\mathrm{sw}_{\mathfrak{p}} = \mathfrak{p} + (1-\mathfrak{p})\sqrt{\varepsilon} \leq \sqrt{\varepsilon} + \sqrt{\varepsilon}(1-\sqrt{\varepsilon}) = 2\sqrt{\varepsilon} - \varepsilon$. Therefore, since $\mathfrak{opt} = 1$, we conclude that for the game defined above, the social welfare of every additive ε-Nash equilibrium is at most $2\sqrt{\varepsilon} - \varepsilon$ times the optimal social welfare. This completes the proof of the second part (lower bound) of Theorem 4.1.

4.7 Win-lose games with $\varepsilon \geq \frac{1}{2}$

We note that for the class of win-lose games, one can easily show the following stronger bound.

Theorem 4.5 *For any win-lose bimatrix game with values in {0,1} and any $\varepsilon \in [\frac{1}{2}, 1]$, we can find in polynomial-time an additive ε-Nash equilibrium with optimal social welfare.*

Proof. Consider a win-lose bimatrix game in which every entry of the pair of the payoffs of the players belongs to the set $\{\{0,0\},\{1,0\},\{0,1\}\}$. We exclude the trivial pure Nash equilibrium $\{1,1\}$, it is easy to see that this equilibrium is also an optimum strategy profile. If (i,j) is an optimal strategy profile, then we know that except from the trivial case $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$ that $\rho = 1$, for the case of $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$, (4.5) implies $C_{ic} = 1$ so $R_{ic} = 0$, (4.6) implies $R_{ij} = 1$ so $C_{ij} = 0$. By Lemma 4.4, for any $\varepsilon \in [\frac{1}{2}, 1]$ there is an additive ε-Nash equilibrium for $\mathfrak{p} = \varepsilon$, so in this case the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$ is an additive

ε-Nash equilibrium with

$$\begin{aligned} \rho &= \frac{\mathtt{opt}}{\mathrm{sw}} = \frac{R_{ij} + C_{ij}}{\mathfrak{p}(R_{ij} + C_{ij}) + (1 - \mathfrak{p})(R_{ic} + C_{ic})} \\ &= \frac{1}{\mathfrak{p} + (1 - \mathfrak{p})} = 1. \end{aligned}$$

□

4.8 Approximation with $\varepsilon < \frac{1}{2}$

The analysis of the case $\varepsilon < \frac{1}{2}$ is more complicated and our results are not as tight as those for the case $\varepsilon \geq \frac{1}{2}$. One important reason why this case is more challenging is that for $\varepsilon < \frac{1}{2}$, we know that we have to consider large support size of the strategies. This follows from [28], who showed that for $\varepsilon < \frac{1}{2}$, to find an additive ε-Nash equilibrium the support needs to be of size logarithmic in the number of strategies available to the players.

We begin with a general transformation that takes an arbitrary additive ε^*-Nash equilibria with arbitrary social welfare and outputs an additive ε-Nash equilibrium, $\varepsilon^* < \varepsilon$, with social welfare at least a constant fraction the optimal social welfare. This is achieved by considering an appropriate mixture of a strategy profile with the optimal social welfare and an additive ε^*-Nash equilibrium. We also show that our transformation runs in polynomial-time, and thus if there is a polynomial-time algorithm finding an additive ε^*-Nash equilibrium then our scheme can find in polynomial-time an additive ε-Nash equilibrium, $\varepsilon^* < \varepsilon$, with social welfare at least a constant fraction the optimal social welfare. Next, we will analyze the special case where the social welfare is greater or equal to $\frac{2-3\varepsilon}{1-\varepsilon}$, when we find additive ε-Nash equilibria with high social welfare.

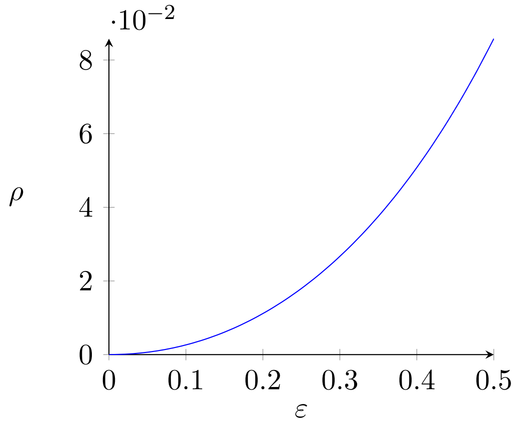

Figure 4.2: Bound for $(1-\sqrt{1-\varepsilon})^2$ as a function of ε, as in Theorem 4.2; $(1-\sqrt{1-\varepsilon})^2 \approx 0.0858$ for $\varepsilon = \frac{1}{2}$.

4.9 Reducing social welfare

As mentioned earlier, if $\varepsilon < \frac{1}{2}$ then we cannot hope to find an additive ε-Nash equilibrium with constant size support, which is the approach we used in Section 4.4. However, we will show that using an existing additive ε^*-Nash equilibrium, $\varepsilon^* < \varepsilon$, with an arbitrary social welfare, we can construct an additive ε-Nash equilibrium with social welfare that is at least a constant times optimal, to conclude Theorem 4.2. We begin with the following key lemma.

Lemma 4.6 *Let $0 \le \varepsilon^* < 1$, and $\varepsilon^* < \varepsilon < 1$. Let (x^*, y^*) be the strategy profile of an additive ε^*-Nash equilibrium. Then, for $\mathfrak{p} = 1 - \sqrt{(1-\varepsilon)/(1-\varepsilon^*)}$, the strategy profile $(\mathfrak{p}i + (1-\mathfrak{p})x^*, \mathfrak{p}j + (1-\mathfrak{p})y^*)$ is an additive ε-Nash equilibrium with the social welfare* $\mathrm{sw} \ge \mathfrak{p}^2 \cdot \mathfrak{opt}$.

Proof. Since (x^*, y^*) is a strategy profile of an additive ε^*-Nash equilibrium, the maximum incentive to deviate for any player in the strategy profile (x^*, y^*) is ε^*. Therefore, since under strategies (i, j), (i, y^*), (x^*, j) no player can improve its payoff by more than 1, we obtain that if the players play the strategy profile $(\mathfrak{p}i + (1-\mathfrak{p})x^*, \mathfrak{p}j + (1-\mathfrak{p})y^*)$, then the maximum incentive to deviate for any player is upper bounded by the following:

$$(1-\mathfrak{p})^2\varepsilon^* + \mathfrak{p}^2 + \mathfrak{p}(1-\mathfrak{p}) + \mathfrak{p}(1-\mathfrak{p}) = 1 - (1-\mathfrak{p})^2(1-\varepsilon^*).$$

Hence, to ensure that this strategy is an additive ε-Nash equilibrium for $0 \le \varepsilon^* < \varepsilon < 1$, we set $\mathfrak{p} = 1 - \sqrt{(1-\varepsilon)/(1-\varepsilon^*)}$. It is easy to check that $0 \le \mathfrak{p} \le 1$.

Next, we can bound the social welfare $\mathrm{sw} = \mathfrak{p}^2(R_{ij}+C_{ij})+(1-\mathfrak{p})^2 x^{*T}(R+C)y^* \ge \mathfrak{p}^2(R_{ij}+C_{ij}) = \mathfrak{p}^2 \cdot \mathtt{opt}$. $\square$

Proof of Theorem 4.2: We choose $\varepsilon^* = 0$ in Lemma 4.6 (here we use Nash theorem [45] to guarantee the existence of an exact Nash equilibrium (x^*, y^*)) to ensure that one can use the strategy profile $(\mathfrak{p}i + (1-\mathfrak{p})x^*, \mathfrak{p}j + (1-\mathfrak{p})y^*)$ with $\mathfrak{p} = 1 - \sqrt{1-\varepsilon}$ to obtain an additive ε-Nash equilibrium with $\mathrm{sw} \ge \mathfrak{p}^2 \cdot \mathtt{opt} = (1-\sqrt{1-\varepsilon})^2 \cdot \mathtt{opt}$. $\square$

Our construction above can be trivially transformed into a polynomial-time algorithm, assuming that we have at hand a polynomial-time algorithm for finding an additive ε^*-Nash equilibrium in any bimatrix game. This proves Theorem 4.3 with $\zeta_{\varepsilon,\varepsilon^*} = (1 - \sqrt{\frac{1-\varepsilon}{1-\varepsilon^*}})^2$. Since the best currently known value for ε^* is 0.3393 [51], this approach works (currently) only for $\varepsilon > 0.3393$.

4.10 Analysis of the case $\mathtt{opt} \ge \frac{2-3\varepsilon}{1-\varepsilon}$

We consider a special case, when $\mathtt{opt} \ge \frac{2-3\varepsilon}{1-\varepsilon}$, for which we can construct additive ε-Nash equilibria with high social welfare. We will show in Theorem 4.8 that there is a good additive ε-Nash equilibrium that has a constant size support and high social welfare. This result is complemented by Theorem 4.9 that shows that if $\mathfrak{opt} < \frac{2-3\varepsilon}{1-\varepsilon}$, then an additive ε-Nash equilibrium may require a logarithmic size support.

We begin with the case $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$ (the case $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$ is symmetric).

Lemma 4.7 *Let $\varepsilon \in [0, \frac{1}{2})$, $R_{rj} - R_{ij} < \varepsilon$, $C_{ic} - C_{ij} > \varepsilon$, and $\mathfrak{p} = \frac{1-\varepsilon}{-1+2\varepsilon+2R_{ij}(1-\varepsilon)}$. If $\mathfrak{opt} \geq \frac{2-3\varepsilon}{1-\varepsilon}$ then the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$ is an additive ε-Nash equilibrium with social welfare greater than $\frac{2-4\varepsilon+\varepsilon^2}{2-3\varepsilon} \cdot \mathfrak{opt} = (1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}) \cdot \mathfrak{opt}$.*

Proof. We first show that $\mathfrak{p}$ is well defined with $0 \leq \mathfrak{p} \leq 1$. Since $C_{ic} - C_{ij} > \varepsilon$, we get $C_{ij} < 1 - \varepsilon$. Thus, if $\mathfrak{opt} \geq \frac{2-3\varepsilon}{1-\varepsilon}$, then $\mathfrak{opt} = R_{ij} + C_{ij}$ yields $R_{ij} \in (\frac{1-\varepsilon-\varepsilon^2}{1-\varepsilon}, 1]$ and $C_{ij} \in [\frac{1-2\varepsilon}{1-\varepsilon}, 1-\varepsilon)$. Hence, $-1 + 2\varepsilon + 2R_{ij}(1-\varepsilon) > -1 + 2\varepsilon + 2(1 - \varepsilon - \varepsilon^2) \geq 1 - \varepsilon$, and thus $\mathfrak{p}$ is well defined.

Next, we prove that the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$ is an additive ε-Nash equilibrium. Let b be the best-response strategy of the row player to the strategy $(\mathfrak{p}j + (1-\mathfrak{p})c)$ of the column player. Then the incentive of the row player to deviate from strategy i is:

$$\mathfrak{p}R_{bj} + (1-\mathfrak{p})R_{bc} - \mathfrak{p}R_{ij} - (1-\mathfrak{p})R_{ic} \leq 1 - \mathfrak{p}R_{ij},$$

and the incentive of the column player to deviate is:

$$\begin{aligned} C_{ic} - \mathfrak{p}C_{ij} - (1-\mathfrak{p})C_{ic} &= \mathfrak{p}(C_{ic} - C_{ij}) \\ \leq \ \mathfrak{p}\left(1 - \left(\frac{2-3\varepsilon}{1-\varepsilon} - R_{ij}\right)\right) &= \mathfrak{p}\left(R_{ij} - \frac{1-2\varepsilon}{1-\varepsilon}\right). \end{aligned}$$

Here we use the facts that c is the best-response strategy of the column player to the strategy i of the row player, and that $C_{ic} \leq 1$ and $C_{ij} = \mathfrak{opt} - R_{ij} \geq \frac{2-3\varepsilon}{1-\varepsilon} - R_{ij}$.

Our choice of $\mathfrak{p}$ ensures that $\mathfrak{p}\left(R_{ij} - \frac{1-2\varepsilon}{1-\varepsilon}\right) = 1-\mathfrak{p}R_{ij} = \frac{-1+2\varepsilon+R_{ij}(1-\varepsilon)}{-1+2\varepsilon+2R_{ij}(1-\varepsilon)}$, which takes the maximum at $R_{ij} = 1$. Therefore $1 - \mathfrak{p}R_{ij} \leq \frac{-1+2\varepsilon+(1-\varepsilon)}{-1+2\varepsilon+2(1-\varepsilon)} = \varepsilon$, what implies that the strategy profile $(i, \mathfrak{p}j+(1-\mathfrak{p})c)$ is an additive ε-Nash equilibrium. This yields the following lower bound for the social welfare sw of the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$:

$$\begin{aligned} \frac{\mathtt{opt}}{\mathrm{sw}} &\leq \frac{1+\frac{1-2\varepsilon}{1-\varepsilon}}{\mathfrak{p}R_{ij}+\frac{1-2\varepsilon}{1-\varepsilon}} = \frac{1+\frac{1-2\varepsilon}{1-\varepsilon}}{\frac{R_{ij}(1-\varepsilon)}{-1+2\varepsilon+2R_{ij}(1-\varepsilon)}+\frac{1-2\varepsilon}{1-\varepsilon}} \\ &\leq \frac{1+\frac{1-2\varepsilon}{1-\varepsilon}}{1-\varepsilon+\frac{1-2\varepsilon}{1-\varepsilon}} = \frac{2-3\varepsilon}{2-4\varepsilon+\varepsilon^2}. \end{aligned}$$

□

With Lemma 4.7 at hand, we can prove the following.

Theorem 4.8 *Let $\varepsilon \in [0, \frac{1}{2})$ and $\mathtt{opt} \geq \frac{2-3\varepsilon}{1-\varepsilon}$. Then one can find in polynomial-time an additive ε-Nash equilibrium with constant support size and with social welfare at least $(1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}) \cdot \mathtt{opt} \geq 0.5 \cdot \mathtt{opt}$.*

Proof. We consider three cases:

- If $R_{rj} - R_{ij} \leq \varepsilon$ and $C_{ic} - C_{ij} \leq \varepsilon$, then the strategy profile (i, j) is an additive ε-Nash equilibrium with $\mathrm{sw} = \mathtt{opt}$.

- If $R_{rj} - R_{ij} \geq \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$ (the $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} \geq \varepsilon$ is symmetric), then $\mathtt{opt} = R_{ij} + C_{ij} < (R_{rj} - \varepsilon) + (C_{ic} - \varepsilon) \leq 2(1-\varepsilon)$. But is impossible if at the same time $\varepsilon < \frac{1}{2}$ and $\mathtt{opt} \geq \frac{2-3\varepsilon}{1-\varepsilon}$, and therefore this case cannot happen.

- Finally, if $R_{rj} - R_{ij} < \varepsilon$ and $C_{ic} - C_{ij} > \varepsilon$ (the case $R_{rj} - R_{ij} > \varepsilon$ and $C_{ic} - C_{ij} < \varepsilon$ is symmetric), then by Lemma 4.7, the strategy profile $(i, \mathfrak{p}j + (1-\mathfrak{p})c)$ with $\mathfrak{p} = \frac{1-\varepsilon}{-1+2\varepsilon+2R_{ij}(1-\varepsilon)}$, is an additive ε-Nash equilibrium with social welfare $\mathrm{sw} \geq (1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}) \cdot \mathtt{opt}$.

The bound $(1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}) \cdot \mathtt{opt} \geq \frac{1}{2} \cdot \mathtt{opt}$ follows from the fact that in the interval $\varepsilon \in [0, \frac{1}{2}]$, function $1 - \frac{\varepsilon(1-\varepsilon)}{2-3\varepsilon}$ is non-increasing in ε, and hence it is minimized at $\varepsilon = \frac{1}{2}$ with the value $\frac{1}{2}$.

All required strategies can be found in polynomial-time. $\square$

Theorem 4.8 ensures that if $\mathtt{opt} \geq \frac{2-3\varepsilon}{1-\varepsilon}$ and $\varepsilon < \frac{1}{2}$, then we can create an additive ε-Nash equilibrium with social welfare greater than or equal to $\frac{1}{2}\mathtt{opt}$, which is a superior upper bound to the general case from Theorem 4.2.

Lower bound. We can prove also a lower bound that for any $\varepsilon \leq \frac{1}{2}$, if $\mathtt{opt} = \frac{2-3\varepsilon}{1-\varepsilon}$ then for any $\hat{\varepsilon} < \varepsilon$, we may need support of size $\Omega(\log n)$ to construct an additive $\hat{\varepsilon}$-Nash equilibrium.

Theorem 4.9 *Let $\varepsilon \leq \frac{1}{2}$. There exists a bimatrix game (R, C) in $[0,1]^{n \times n}$ for which the maximum sum of the payoffs of the players is $\mathtt{opt} = \frac{2-3\varepsilon}{1-\varepsilon}$, and for any $\hat{\varepsilon} < \varepsilon$, any additive $\hat{\varepsilon}$-Nash equilibrium requires logarithmic support.*

Proof. Let $k = \log n - 2 \log \log n$. Let (R, C) be the two payoff matrices in $[0,1]^{n \times n}$ in which every entry is chosen independently at random from the set $\{(1, \frac{1-2\varepsilon}{1-\varepsilon}), (0,1)\}$. We consider the row player; the case of the column player is analogous. We will show that with high probability, for any k columns in the payoff matrix of the column player, there is at least one row that has all 1s in these k columns.

Fix any set of k columns. The probability that a single row has at least one 0 in these k columns is $1 - 2^{-k}$. Thus, the probability that every row has at least one 0 in these k columns is $(1 - 2^{-k})^n$. Hence, the probability that there is a set of k columns for which all rows have at least one 0 in these k columns is at most $\binom{n}{k}(1 - 2^{-k})^n$. Since our choice of k yields

$\binom{n}{k}(1-2^{-k})^n \ll 1$, we conclude that with high probability, for every set of k columns there is at least one row that has all 1s in these k columns. Analogous arguments hold for the column player. Let us condition on the two events and assume that for every set of k columns in the payoff matrix of the row player there is a row that has all 1s in these columns, and that for every set of k rows in the payoff matrix of the column player there is a column that has all 1s in these rows.

Let us assume that there is an additive $\hat{\varepsilon}$-Nash equilibrium (x^*, y^*) for some $\hat{\varepsilon} < \varepsilon \le \frac{1}{2}$, with the support of size k. Let $p = \sum_{\ell,m} x^*_\ell y^*_m$, where the sum is over all pairs (ℓ, m), $1 \le \ell, m \le n$, such that $(R_{\ell m}, C_{\ell m}) = (1, \frac{1-2\varepsilon}{1-\varepsilon})$. p is the probability that the players play the strategy profile $(1, \frac{1-2\varepsilon}{1-\varepsilon})$ in the additive $\hat{\varepsilon}$-Nash equilibrium, and $1-p$ is the probability that the players play the strategy profile $(0,1)$ in the additive $\hat{\varepsilon}$-Nash equilibrium. The expected payoff of the row player is p, and the expected payoff of the column player is $p\left(\frac{1-2\varepsilon}{1-\varepsilon}\right) + (1-p)$.

Since (x^*, y^*) is an additive $\hat{\varepsilon}$-Nash equilibrium, $p + \hat{\varepsilon} \ge 1$ for the row player, and thus $p > 1 - \varepsilon$. Hence, the expected payoff of the column player is $p\left(\frac{1-2\varepsilon}{1-\varepsilon}\right) + (1-p) = 1 - \frac{p\varepsilon}{1-\varepsilon} < 1 - \varepsilon < 1 - \hat{\varepsilon}$. But this contradicts the condition for the column player in the assumption that (x^*, y^*) is an additive $\hat{\varepsilon}$-Nash equilibrium.

$\square$

Chapter 5

Plutocratic and egalitarian ε-NE

The fundamental problem in game theory is to study properties of equilibria in non-cooperative games. By the classical result of Nash [45], we know that every finite game has at least one Nash equilibrium. However, in many natural scenarios one is not only interested in finding or characterizing any equilibrium, but in fact, one wants to find equilibria that will have some desirable properties, one wants to find "best" or "fairest" equilibria. While we understand quite well various properties of the set of all Nash equilibria, if we consider also the quality of a solution sought and aim at characterizing the "best" Nash equilibria, then our knowledge is rather limited.

The situation is even more challenging if we take into account the computational complexity of the task. For example, even in two-player games, the problem of finding an arbitrary Nash equilibrium is known to be PPAD-complete and the problem of finding a Nash equilibrium with almost any additional constraint is NP-hard, and thus, it is likely to be computationally even more difficult (see, e.g., [30, 12, 43]). In fact, it is even NP-hard to approximate (to any positive ratio) the maximum social welfare, or the maximum egalitarian payoffs, or the maximum plutocratic payoffs, obtained in an exact Nash equilibrium, even in symmetric two-player games [12, Corollaries 6–8].

In this chapter we study the fundamental problem of *computing additive ε-Nash equilibria* that are *close to the "best" Nash equilibrium* in bimatrix games. Our focus is on the analysis of *"best"* or *"approximately best"* additive ε-Nash equilibria in a two-player game for a fixed ε, for the regime when we know that we can find an additive ε-Nash equilibrium. The central question studied in this chapter is if we allow the players up to ε loss to deviate from the best-response strategy, whether we can find a stable strategy profile (an additive ε-Nash equilibrium) that guarantees the players benefits close to the optimum in any Nash equilibrium? Results of this chapter also appeared in [18].

There have been some investigations aiming to efficiently find additive ε-Nash equilibrium that also approximate well the *social welfare* (the total payoff of the players), see Chapter 4. However in this chapter, we will consider equilibria focusing on other objectives: instead of aiming to maximize the social welfare, or, in short, expected payoff of the players, we will study the scenarios where one wants to maximize the smallest of the payoffs, or that one wants to maximize the largest of the payoffs. That is, we will consider two natural settings to describe the notion of the "best Nash equilibrium":

- *plutocratic Nash equilibrium*—that *maximizes the expected* *<u>maximum</u>* *of the payoffs* of the players;

- *egalitarian Nash equilibrium*—that *maximizes the expected* *<u>minimum</u>* *of the payoffs* of the players.

We provide a description of the quality of additive ε-Nash equilibria that approximate best plutocratic and egalitarian Nash equilibrium for any $\varepsilon \geq \frac{3-\sqrt{5}}{2} \approx 0.382$. Furthermore, we match our existential results with polynomial-time algorithms to find additive ε-Nash equilibria that approximate the best Nash equilibrium. The results are obtained by first showing the existence of such additive ε-Nash equilibria adapting the method due to Daskalakis et al. [23] and by carefully maintaining the tradeoff between the payoffs of the players. Once the existence of an appropriate additive ε-Nash equilibria is given, we apply an LP-based approach to find it.

Observe that our analysis assumes that $\varepsilon \geq \frac{3-\sqrt{5}}{2} \approx 0.382$, which covers almost the entire regime when we know that we can find an additive ε-Nash equilibrium in polynomial-time. Indeed, the smallest value of ε for which we know how to find an additive ε-Nash equilibrium in polynomial-time is $\varepsilon \approx 0.3393$ [51].

The tradeoff between additive ε-Nash equilibria and the quality of the solution found has been studied in the past, though in this context, the main focus of the research was on the goal of finding additive ε-Nash equilibria that approximate the optimal social welfare (the expected total payoff of the players). There have been two strands of these investigations, one comparing the social welfare to the optimal social welfare of the game and one comparing it to the optimal social welfare in any Nash equilibrium.

It is known that a Nash equilibrium can be arbitrarily far from the optimal social welfare in a bimatrix game, see, e.g., Chapter 4. Motivated by this result, there has been recent study aiming to find an additive ε-Nash equilibrium with the social welfare close to the best possible (optimal social welfare of the game, not necessarily in a Nash equilibrium). It was shown in Chapter 4 that for every fixed $\varepsilon > 0$, every bimatrix game (with values in $[0, 1]$) has an additive ε-Nash equilibrium with the social welfare at least a constant factor of the optimum. For example, for any $\varepsilon \geq \frac{1}{2}$, there is always an additive ε-Nash equilibrium whose social welfare is at least $2\sqrt{\varepsilon} - \varepsilon \geq 0.914$ times the optimal social welfare, and this bound is tight. Furthermore, these results are algorithmic, and for every fixed $0 \leq \varepsilon^* < \varepsilon$, if one can find an additive ε^*-Nash equilibrium in polynomial-time, then one can find in polynomial-time an additive ε-Nash equilibrium with the social welfare at least a constant factor of the optimum.

There has been also some research aiming to efficiently find an additive ε-Nash equilibrium that *approximates well the social welfare in any Nash equilibrium*. (Observe that if an additive ε-Nash equilibrium approximates well the optimal social welfare, then it also approximates well the social welfare in any Nash equilibrium.) It has been noted that the quasi-polynomial-time algorithm for approximating Nash equilibrium by Lipton et al. [42] not only finds an additive ε-Nash equilibrium for arbitrary $\varepsilon > 0$, but also the social welfare of the equilibrium found is an ε-approximation of the social welfare in any Nash equilibrium. In other words, in time $n^{\mathcal{O}(\log n/\varepsilon^2)}$ we can find an arbitrarily good additive ε-Nash equilibrium with social welfare near to the best Nash equilibrium. (Further, we note that it is straightforward to extend the quasi-polynomial-time algorithm from [42] to find an additive ε-Nash equilibrium whose plutocratic (or egalitarian) payoff is no more than ε smaller than the maximum plutocratic (or egalitarian, respectively) payoff of a Nash equilibrium in the game.) While this result raised a hope that it may be possible to extend it to design a polynomial-time algorithm, recent hardness results [4, 8, 34] showed that it is unlikely. Braverman et al. [8] showed that assuming the deterministic Exponential Time Hypothesis, there is a constant $\varepsilon > 0$ such that any algorithm for finding an additive ε-Nash equilibrium whose social welfare is at least $(1-\varepsilon)$ times the optimal social welfare of a Nash equilibrium of the game, requires $n^{\Omega(\log n)}$ time (see also Hazan and Krauthgamer [34] and Austrin et al. [4] for related results that assume hardness of finding a large planted clique). These hardness results show that it is very unlikely to obtain a polynomial-time approximation scheme that for every positive constants ε and ε' would construct in polynomial-time an additive ε-Nash equilibrium whose social welfare is at least $(1-\varepsilon')$ times the optimal social welfare of a Nash equilibrium of the game. (Austrin et al. [4] showed that also many similar variants of the bi-criteria approximation are similarly hard.) Also, in [24] Deligkas et al. give inapproximability results for approximate Nash equilibria that are δ close to the best social welfare achievable by an ε-NE. On the other hand, the results from Chapter 4 can be combined with the polynomial-time algorithm finding an additive ε^*-Nash equilibrium in any bimatrix game with $\varepsilon^* \approx 0.3393$ [51], to obtain a polynomial-time algorithm that for any $\varepsilon > \varepsilon^*$ finds an additive ε-Nash equilibrium whose social welfare is at least $(1-\sqrt{\frac{1-\varepsilon}{1-\varepsilon^*}})^2$ times the optimal social welfare of a Nash equilibrium of the game; for $\varepsilon \geq \frac{1}{2}$, the approximation bound can be made at least $2\sqrt{\varepsilon}-\varepsilon \geq 0.914$ times the optimal social welfare.

5.1 Preliminaries

In this chapter, our goal is to find additive ε-NE that are close to a "best" NE. We consider two different and very natural versions of a best NE: a NE that maximizes the *maximum* of the payoffs of the two players (a *plutocratic* NE) and a NE that maximizes the *minimum* of the payoffs (an *egalitarian* NE) [32].

Definition 5.1 *A* ***plutocratic NE*** *is a NE* (x^*, y^*) *such that:*

$$u^*_{\max} \stackrel{def}{=} \max\{x^{*T}Ry^*, x^{*T}Cy^*\} = \max\{\max\{x^TRy, x^TCy\} : (x, y) \text{ is a NE}\}.$$

Definition 5.2 *An* ***egalitarian NE*** *is a NE* (x^*, y^*) *such that:*

$$u^*_{eg} \stackrel{def}{=} \min\{x^{*T}Ry^*, x^{*T}Cy^*\} = \max\{\min\{x^TRy, x^TCy\} : (x, y) \text{ is a NE}\}.$$

In other words, a plutocratic NE is a NE that maximizes the maximum payoff of the two players, and an egalitarian NE is a NE that maximizes the minimum payoff of the two players.

We extend these notions to additive ε-Nash equilibria.

Definition 5.3 *An* ***(additive)*** ρ***-plutocratic additive*** ε***-NE*** *is an additive ε-NE (x, y) such that $u^*_{\max} - \max\{x^TRy, x^TCy\} \leq \rho$. If a strategy profile (x, y) is a 0-plutocratic additive ε-NE then we simply say that it is a* ***plutocratic additive*** ε***-NE****.*

Definition 5.4 *An* ***(additive)*** ρ***-egalitarian additive*** ε***-NE*** *is an additive ε-NE (x, y) such that $u^*_{eg} - \min\{x^TRy, x^TCy\} \leq \rho$.*

A ***relative*** ρ***-egalitarian additive*** ε***-NE*** *is an additive ε-NE (x, y) such that: $\rho \cdot u^*_{eg} \leq \min\{x^TRy, x^TCy\}$.*

Before we present our results, we recall a simple lemma by Austrin et al. [4] which we use frequently in our analysis. (Since this claim was stated without proof in [4], for the sake of completeness we include a simple proof.)

Lemma 5.5 [4] *Let (R, C) be a bimatrix game and let (x^*, y^*) be a NE with expected payoffs u_R^* and u_C^* for the row and the column player, respectively.*

Then there is a strategy profile (x, j), such that $x^T Re_j \geq u_R^*$ and j is a pure best-response to x (i.e., $x^T Ce_j = u_C^*$).

Similarly, there is a strategy profile (i, y), such that $e_i^T Cy \geq u_C^*$ and i is a pure best-response to y (i.e., $e_i^T Ry = u_R^*$).

Proof. We only prove the first claim; the proof of the second claim is analogous.

Since u_R^* is the expected payoff of the row player in the Nash equilibrium (x^*, y^*), we know that $u_R^* = x^{*T} Ry^* = \sum_{j=1}^{n} (x^{*T} Re_j) y^*(j)$. Therefore, there must be at least one column $j \in supp(y^*)$ such that $x^{*T} Re_j \geq u_R^*$. Further, since $j \in supp(y^*)$, j is a best-response strategy for the column player, with $x^{*T} Ce_j = u_C^*$. □

5.2 Approximate Plutocratic NE

We begin our analysis with the problem of finding good additive ε-NE that are close to plutocratic NE. We consider two cases, depending on whether $\varepsilon \geq \frac{1}{2}$ or not. In the first case, when $\varepsilon \geq \frac{1}{2}$, we give a polynomial-time algorithm that finds a plutocratic additive $\frac{1}{2}$-NE, that is, an additive $\frac{1}{2}$-NE whose maximum payoff is at least as large as the maximum payoff of a plutocratic NE. For smaller values of ε, $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, we give a polynomial-time algorithm that finds an $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE. Note that the function $\frac{1-2\varepsilon}{1-\varepsilon} = 1 - \frac{\varepsilon}{1-\varepsilon}$ decreases monotonically from $\frac{3-\sqrt{5}}{2}$ at $\varepsilon = \frac{3-\sqrt{5}}{2}$ down to 0 at $\varepsilon = \frac{1}{2}$.

5.2.1 The First Case: $\varepsilon \geq \frac{1}{2}$

In this subsection, we give a construction of an additive $\frac{1}{2}$-NE (which is of course an additive ε-NE for all $\varepsilon \geq \frac{1}{2}$) whose maximum payoff is at least as large as the maximum payoff of a plutocratic NE. We then argue that this construction can be turned into a polynomial-time algorithm that computes such a plutocratic additive $\frac{1}{2}$-NE.

Let (x^*, y^*) be a plutocratic NE. We assume, without loss of generality, that $u^*_{\max} = x^{*T} R y^*$. Then, by Lemma 5.5, there is a pure strategy j such that: $x^{*T} R e_j \geq u^*_{\max}$, and for all i, we have $x^{*T} C e_j \geq x^{*T} C e_i$.

Let r be a pure best-response strategy of the row player to the pure strategy j of the column player. We claim that the strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ is an additive $\frac{1}{2}$-NE, and the maximum payoff is at least as large as the maximum payoff of a plutocratic NE.

This construction is very similar to the construction of Daskalakis et al. [23] and the proof that it gives an additive $\frac{1}{2}$-NE (that we present for completeness below) is analogous to theirs.

Lemma 5.6 *The strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ is an additive $\frac{1}{2}$-NE.*

Proof. Let c be the best-response strategy of the column player to the strategy $(\frac{1}{2}x^* + \frac{1}{2}r)$ of the row player. The maximum incentive for the row player to deviate is equal to her payoff in her best-response (which is r)

minus her current payoff, that is,

$$e_r^T Re_j - \frac{1}{2}x^{*T}Re_j - \frac{1}{2}e_r^T Re_j \leq \frac{1}{2}e_r^T Re_j \leq \frac{1}{2}.$$

The first inequality holds since $x^{*T}Re_j \geq 0$ and the second inequality holds since $e_r^T Re_j \leq 1$.

Similarly, the maximum incentive for the column player to deviate is

$$\begin{aligned}\frac{1}{2}x^{*T}Ce_c + \frac{1}{2}e_r^T Ce_c - \frac{1}{2}x^{*T}Ce_j - \frac{1}{2}e_r^T Ce_j& \\ \leq \frac{1}{2}e_r^T Ce_c - \frac{1}{2}e_r^T Ce_j \leq \frac{1}{2}.&\end{aligned}$$

The first inequality holds since j is the best-response of the column player to the strategy x^* of the row player, so $\frac{1}{2}x^{*T}Ce_c - \frac{1}{2}x^{*T}Ce_j \leq 0$. The second inequality follows from $e_r^T Ce_c \leq 1$ and $e_r^T Ce_j \geq 0$. These two bounds above show that the strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ is an additive $\frac{1}{2}$-Nash equilibrium.

$\square$

Lemma 5.7 *The maximum payoff of the strategy profile* $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ *is at least as large as the maximum payoff of a plutocratic NE.*

Proof. First observe that since $e_r^T Re_j \geq x^{*T}Re_j \geq u^*_{\max}$, we also have $(\frac{1}{2}x^* + \frac{1}{2}e_r)^T Re_j = \frac{1}{2}x^{*T}Re_j + \frac{1}{2}e_r^T Re_j \geq \frac{1}{2}u^*_{\max} + \frac{1}{2}u^*_{\max} = u^*_{\max}$. Therefore,

$$\begin{aligned}u^*_{\max} - \max\{(\frac{1}{2}x^* + \frac{1}{2}e_r)^T Re_j, (\frac{1}{2}x^* + \frac{1}{2}e_r)^T Ce_j\}& \\ \leq u^*_{\max} - \max\{u^*_{\max}, (\frac{1}{2}x^* + \frac{1}{2}e_r)^T Ce_j\} \leq 0.&\end{aligned}$$

$\square$

Algorithm 1

1. For every j, solve the following two LPs:

$$LP1: \quad \text{maximize } u_R \qquad\qquad LP2: \quad \text{maximize } u_C$$

$$x^T R e_j \geq u_R \qquad\qquad e_j^T C y \geq u_C$$

$$\forall i, \quad x^T C e_j \geq x^T C e_i \qquad\qquad \forall i, \quad e_j^T R y \geq e_i^T R y$$

2. Among all feasible choices of j in LP1, find a solution (x, j) that maximizes u_R.

   Then, find the best-response strategy r of the row player to the strategy j of the column player.

3. Among all feasible choices of j in LP2, find a solution (j, y) that maximizes u_C.

   Then, find the best-response strategy c of the column player to the strategy j of the row player.

4. Let u_R and u_C be the maximum values in Steps 2 and 3, respectively.

   If $u_R \geq u_C$ then return strategy profile $(\frac{1}{2}x+\frac{1}{2}r, j)$; otherwise return strategy profile $(j, \frac{1}{2}y + \frac{1}{2}c)$.

Now we argue that the strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ described above can be computed in polynomial-time, which is the main result of this subsection.

Theorem 5.8 *There is a polynomial-time algorithm that — given a bimatrix game (R, C) with values in $[0, 1]$ as input — computes a plutocratic additive $\frac{1}{2}$-NE.*

Proof. We will show that Algorithm 1 presented above returns a plutocratic additive $\frac{1}{2}$-NE. Let (x^*, y^*) be a NE with the expected payoffs u_R^* and u_C^* for the row player and for the column player, respectively. By Lemma 5.5, there is a pure strategy j, such that the strategy profile (x^*, j) is a feasible solution in the LP1, because $x^{*T} Re_j \geq u_R^*$, and for all i, we have $x^{*T} Ce_j \geq x^{*T} Ce_i$. Therefore, the value u_R found in Step 2 satisfies $u_R \geq u_R^*$.

Similarly, there is a pure strategy j such that the strategy profile (j, y^*) is a feasible solution in the LP2 since, by Lemma 5.5, $e_j^T Cy^* \geq u_C^*$, and for all i, we have $e_j^T Ry^* \geq e_i^T Ry^*$. Therefore, the value u_C found in Step 3 satisfies $u_C \geq u_C^*$. The claims above mean that the values u_R and u_C found in Steps 2 and 3 satisfy $\max\{u_R, u_C\} \geq u_{\max}^* = \max\{u_R^*, u_C^*\}$.

Let us assume, without loss of generality, that $u_R \geq u_C$ and therefore the algorithm returns the strategy profile $(\frac{1}{2}x + \frac{1}{2}r, j)$. Using analogous arguments as those used in the proof of Lemma 5.6, one can see that the strategy profile $(\frac{1}{2}x + \frac{1}{2}r, j)$ is an additive $\frac{1}{2}$-Nash equilibrium. Next, the payoff of the row player is equal to:

$$\frac{1}{2}x^T Re_j + \frac{1}{2}e_r^T Re_j \geq \frac{1}{2}u_R + \frac{1}{2}u_R \geq \frac{1}{2}u_{\max}^* + \frac{1}{2}u_{\max}^* = u_{\max}^*.$$

The first inequality holds since $e_r^T Re_j \geq x^T Re_j \geq u_R$ and the second one follows from $u_R \geq u_{\max}^*$.

This immediately implies that the maximum payoff of the strategy profile $(\frac{1}{2}x + \frac{1}{2}r, j)$ is at least as large as the maximum payoff of a plutocratic NE, because:

$$\begin{aligned} &u_{\max}^* - \max\{(\tfrac{1}{2}x + \tfrac{1}{2}e_r)^T Re_j, (\tfrac{1}{2}x + \tfrac{1}{2}e_r)^T Ce_j\} \\ &\qquad\qquad \leq u_{\max}^* - \max\{u_{\max}^*, (\tfrac{1}{2}x + \tfrac{1}{2}e_r)^T Ce_j\} \leq 0. \end{aligned}$$

Finally, we observe that Algorithm 1 performs $2n$ calls to a linear programming solver in addition to some basic polynomial-time computations, and since linear programming can be solved in polynomial-time, we conclude that Algorithm 1 runs in polynomial-time. $\square$

5.2.2 The Second Case: $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$

Now we consider the case when $\varepsilon < \frac{1}{2}$ and show that if $\varepsilon \geq \frac{3-\sqrt{5}}{2} \approx 0.382$, a similar approach as the one presented in Section 5.2.1 can be applied, though this time the result is weaker: the obtained additive ε-Nash equilibrium is an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE, rather than a plutocratic additive ε-NE, as in Section 5.2.1.

We begin by describing a construction of an appropriate strategy profile, and then we present a polynomial-time algorithm implementing it.

Let (x^*, y^*) be a plutocratic NE. We assume, without loss of generality, that $u^*_{\max} = x^{*T}Ry^*$. We split the analysis into two cases, depending on whether $u^*_{\max} \geq \frac{1-2\varepsilon}{1-\varepsilon}$ or not.

Case 1: $u^*_{\max} \geq \frac{1-2\varepsilon}{1-\varepsilon}$**.**

Let j be a pure best-response strategy of the column player to the strategy x^* of the row player such that $x^{*T}Re_j \geq u^*_{\max}$, whose existence is given by Lemma 5.5. Let r be the best-response strategy of the row player to the pure strategy j of the column player. We claim that the strategy profile $(\frac{1}{2-u^*_{\max}}x^* + (1 - \frac{1}{2-u^*_{\max}})r, j)$ is a plutocratic additive ε-NE.

Lemma 5.9 *The strategy profile* $\left(\frac{1}{2-u^*_{\max}}x^* + (1 - \frac{1}{2-u^*_{\max}})r, j\right)$ *is an additive ε-NE.*

Proof. The maximum incentive for the row player to deviate is upper bounded as follows:

$$e_r^T Re_j - (\frac{1}{2-u^*_{\max}})x^{*T}Re_j - (1 - \frac{1}{2-u^*_{\max}})e_r^T Re_j \leq$$
$$\leq \frac{1}{2-u^*_{\max}}e_r^T Re_j - \frac{u^*_{\max}}{2-u^*_{\max}} \leq \frac{1-u^*_{\max}}{2-u^*_{\max}} \leq \varepsilon.$$

The first inequality follows from $x^{*T}Re_j \geq u^*_{\max}$ and the second one follows from $e_r^T Re_j \leq 1$. To see the third inequality, note that $\frac{1-u^*_{\max}}{2-u^*_{\max}}$ is a decreasing function of $u^*_{\max}$ and thus, it takes the maximum when $u^*_{\max} = \frac{1-2\varepsilon}{1-\varepsilon}$, in which case it is equal to ε.

The maximum incentive for the column player to deviate can be bounded as follows:

$$\frac{1}{2-u^*_{\max}}x^{*T}Ce_c + (1-\frac{1}{2-u^*_{\max}})e_r^T Ce_c - \frac{1}{2-u^*_{\max}}x^{*T}Ce_j$$
$$-(1-\frac{1}{2-u^*_{\max}})e_r^T Ce_j \leq (1-\frac{1}{2-u^*_{\max}})e_r^T Ce_c \leq \frac{1-u^*_{\max}}{2-u^*_{\max}} \leq \varepsilon.$$

The first inequality follows from the fact that since j is the best-response strategy of the column player to the strategy x^* of the row player, we have $\frac{1}{2-u^*_{\max}}x^{*T}Ce_c \leq \frac{1}{2-u^*_{\max}}x^{*T}Ce_j$, and $e_r^T Ce_j \geq 0$. The second inequality holds since $e_r^T Ce_c \leq 1$. These two bounds imply that the strategy profile $(\frac{1}{2-u^*_{\max}}x^* + (1-\frac{1}{2-u^*_{\max}})r, j)$ is an additive ε-Nash equilibrium. □

Lemma 5.10 *The maximum payoff of the strategy profile* $\big(\frac{1}{2-u^*_{\max}}x^* + (1-\frac{1}{2-u^*_{\max}})r, j\big)$ *is at least as large as the maximum payoff of a plutocratic NE.*

Proof. First observe that the payoff of the row player is bounded by the following:

$$\frac{1}{2-u^*_{\max}}x^{*T}Re_j + (1-\frac{1}{2-u^*_{\max}})e_r^T Re_j$$
$$\geq \frac{1}{2-u^*_{\max}}u^*_{\max} + (1-\frac{1}{2-u^*_{\max}})u^*_{\max} = u^*_{\max},$$

where we used the fact that $e_r^T Re_j \geq x^{*T}Re_j \geq u^*_{\max}$.

If we plug in this inequality, then we conclude that the maximum payoff of the strategy profile $(\frac{1}{2-u^*_{\max}}x^* + (1 - \frac{1}{2-u^*_{\max}})r, j)$ is at least as large as the maximum payoff of a plutocratic NE, because:

$$\begin{aligned} & u^*_{\max} - \max\{(\frac{1}{2-u^*_{\max}}x^* + (1-\frac{1}{2-u^*_{\max}})e_r)^T Re_j, \\ & \qquad\qquad (\frac{1}{2-u^*_{\max}}x^* + (1-\frac{1}{2-u^*_{\max}})e_r)^T Ce_j\} \\ \leq\ & u^*_{\max} - \max\{u^*_{\max}, (\frac{1}{2-u^*_{\max}}x^* + (1-\frac{1}{2-u^*_{\max}})e_r)^T Ce_j\} \ \leq\ 0. \end{aligned}$$

□

Case 2: $u^*_{\max} < \frac{1-2\varepsilon}{1-\varepsilon}$**.**

By the definition of NE and by $u^*_{\max} = \max\{x^{*T}Ry^*, x^{*T}Cy^*\}$, there exists a strategy profile (x, y) such that: for all i, we have $e_i{}^T Ry \leq u^*_{\max}$ and $x^T Ce_i \leq u^*_{\max}$.

We claim that if $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, then the strategy profile (x, y) is an additive ε-Nash equilibrium. Indeed, we have,

$$\begin{aligned} \forall i, \quad & e_i{}^T Ry \leq u^*_{\max} < \frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon, \\ \forall i, \quad & x^T Ce_i \leq u^*_{\max} < \frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon, \end{aligned}$$

where we use the fact that if $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, then $\frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon$.

Next we note that the strategy profile (x, y) is an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE because: $u^*_{\max} - \max\{x^T Ry, x^T Cy\} < \frac{1-2\varepsilon}{1-\varepsilon}$.

Now we proceed to describe a polynomial-time algorithm that implements the constructions described above.

Theorem 5.11 *There is a polynomial-time algorithm that — given an ε, $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$ and a bimatrix game (R, C) with value in $[0, 1]$ — computes an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE.*

Algorithm 2

1. For every j, solve the following two LPs:

$$LP1: \quad \text{maximize } u_R \qquad\qquad LP2: \quad \text{maximize } u_C$$

$$x^T R e_j \geq u_R \qquad\qquad e_j^T C y \geq u_C$$

$$\forall i, \quad x^T C e_j \geq x^T C e_i \qquad\qquad \forall i, \quad e_j^T R y \geq e_i^T R y$$

2. For the LP1, among all feasible choices of j find a solution (x, j) that maximizes u_R.

   Then, find the best-response strategy r of the row player to the strategy j of the column player.

3. For the LP2, among all feasible choices of j find a solution (j, y) that maximizes u_C.

   Then, find the best-response strategy c of the column player to the strategy j of the row player.

4. Find $u_{\max} = \max\{u_R, u_C\}$, where u_R and u_C are the maximum values in Steps 2 and 3.

   (a) If $u_{\max} = u_R \geq \frac{1-2\varepsilon}{1-\varepsilon}$, then return strategy profile $(\frac{1}{2-u_{\max}}x + (1 - \frac{1}{2-u_{\max}})r, j)$.

   (b) Else, if $u_{\max} = u_C \geq \frac{1-2\varepsilon}{1-\varepsilon}$, then return strategy profile $(j, \frac{1}{2-u_{\max}}y + (1 - \frac{1}{2-u_{\max}})c)$.

   (c) Otherwise, find and return an arbitrary strategy profile (x, y) such that

$$\forall i, \quad {e_i}^T R y \leq u_{\max},$$

$$\forall i, \quad x^T C e_i \leq u_{\max}.$$

Proof. We prove that if $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, then Algorithm 2 returns an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE. Our proof uses the analysis from Cases 1 and 2 above, and it follows a similar framework as Algorithm 1 (cf. Theorem 5.8).

Similarly as in the proof of Theorem 5.8, we observe that $u_{\max} \geq u^*_{\max}$. Next, we assume, without loss of generality, that $u_{\max} = u_R$. Then, we note that identical arguments as those used in the analysis above imply that the solution returned by the algorithm will be an additive ε-Nash equilibrium. Therefore we only have to show that it is an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE.

We first consider the case when $u_{\max} \geq \frac{1-2\varepsilon}{1-\varepsilon}$, in which case Algorithm 2 returns strategy profile $(\frac{1}{2-u_{\max}}x+(1-\frac{1}{2-u_{\max}})r, j)$. Since $e_r^T Re_j \geq x^T Re_j \geq u_{\max}$, we can bound the payoff of the the row player as follows:

$$\frac{1}{2-u_{\max}}x^T Re_j + (1 - \frac{1}{2-u_{\max}})e_r^T Re_j \geq \frac{1}{2-u_{\max}}u_{\max} \\ + (1 - \frac{1}{2-u_{\max}})u_{\max} = u_{\max} \geq u^*_{\max}.$$

This bound implies that in this case Algorithm 2 returns a strategy profile whose maximum payoff is at least as large as that of a plutocratic NE, because:

$$u^*_{\max} - \max\{(\frac{x}{2-u_{\max}} + (1 - \frac{1}{2-u_{\max}})e_r)^T Re_j, \\ (\frac{x}{2-u_{\max}} + (1 - \frac{1}{2-u_{\max}})e_r)^T Ce_j\} \leq u^*_{\max} - \max\{u_{\max}, \\ (\frac{1}{2-u_{\max}}x + (1 - \frac{1}{2-u_{\max}})e_r)^T Ce_j\} \leq u^*_{\max} \\ - \max\{u^*_{\max}, (\frac{1}{2-u_{\max}}x + (1 - \frac{1}{2-u_{\max}})e_r)^T Ce_j\} \leq 0.$$

Next, we have to consider the case when $u_{\max} < \frac{1-2\varepsilon}{1-\varepsilon}$. In this case, by the definition of NE and the fact that since $u_{\max} \geq u^*_{\max}$, we know that there is a strategy profile (x, y) such that for all i, we have ${e_i}^T Ry \leq u_{\max}$ and $x^T Ce_i \leq u_{\max}$. As we have shown in the above analysis, this strategy profile is an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-plutocratic additive ε-NE.

Finally, we observe that Algorithm 2 performs $2n+1$ calls to linear programming solver, but since linear programming can be solved in polynomial-time, we conclude that Algorithm 2 runs in polynomial-time. $\square$

5.3 Approximate Egalitarian NE

Next, we analyze the problem of finding good additive ε-Nash equilibria that are close to egalitarian equilibria. As in Section 5.2, we consider two cases, depending on whether $\varepsilon \geq \frac{1}{2}$, or $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$. In the first case, we show that it is possible to compute relative $\frac{1}{2}$-egalitarian additive ε-NE in polynomial-time. In the second case, we show that it is possible to compute additive $\frac{1}{2}$-egalitarian additive ε-NE in polynomial-time.

5.3.1 The First Case: $\varepsilon \geq 1/2$

We first describe a construction that yields relative $\frac{1}{2}$-egalitarian additive $\frac{1}{2}$-NE, and then we explain how this construction can be implemented in polynomial-time.

Let (x^*, y^*) be an egalitarian NE. Let u_R^* and u_C^* be the expected payoffs for the row player and for the column player, respectively. We assume, without loss of generality, that $u_{eg}^* = u_C^* = x^{*T} Cy^*$; in other words $u_R^* \geq u_C^*$. Then, by Lemma 5.5, there is a pure strategy $j \in \mathit{supp}(y^*)$ such that: $x^{*T} Re_j \geq u_R^*$ and for all i, we have $u_C^* = x^{*T} Ce_j \geq x^{*T} Ce_i$.

Let r be the best-response strategy of the row player to the strategy j of the column player. We claim that the strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ is a relative $\frac{1}{2}$-egalitarian additive $\frac{1}{2}$-Nash equilibrium. We first note that the proof of Lemma 5.6 showing that this strategy profile is an additive $\frac{1}{2}$-Nash equilibrium can be directly applied here. Therefore, we will focus here on relative egalitarian approximation.

The payoff of the row player is equal to:

$$\frac{1}{2}x^{*T}Re_j + \frac{1}{2}e_r^T Re_j \geq \frac{1}{2}u_R^* + \frac{1}{2}u_R^* \geq u_R^*,$$

where the first inequality holds since $e_r^T Re_j \geq x^{*T}Re_j \geq u_R^*$. Therefore, the strategy profile $(\frac{1}{2}x^* + \frac{1}{2}r, j)$ is a relative $\frac{1}{2}$-egalitarian additive $\frac{1}{2}$-NE because:

$$\begin{aligned}\min\{(\frac{1}{2}x^* + \frac{1}{2}e_r)^T Re_j, (\frac{1}{2}x^* + \frac{1}{2}e_r)^T Ce_j\} \\ \geq \min\{u_R^*, \frac{1}{2}x^{*T}Ce_j\} = \min\{u_R^*, \frac{1}{2}u_{eg}^*\} = \frac{1}{2}u_{eg}^*.\end{aligned}$$

We argue that the above construction can be implemented in polynomial-time.

Theorem 5.12 *There is a polynomial-time algorithm that — given a bi-matrix game (R, C) with values in $[0, 1]$ — computes a relative $\frac{1}{2}$-egalitarian additive $\frac{1}{2}$-NE.*

Proof. We will demonstrate that Algorithm 3 satisfies the promised properties of the theorem.

Let (x^*, y^*) be an egalitarian NE. Let u_R^* and u_C^* be the expected payoffs for the row player and for the column player, respectively. If $u_R^* \geq u_C^* = u_{eg}^*$, then by Lemma 5.5, there is a strategy profile (x^*, j) which is a feasible solution of the LP1 since: $x^{*T}Re_j \geq u_R^* \geq u_C^* = x^{*T}Ce_j$, and for all i, we have $x^{*T}Ce_j \geq x^{*T}Ce_i$.

Algorithm 3

1. For every j, solve the following two LPs:

$$LP1: \quad \text{maximize } x^T C e_j \qquad LP2: \quad \text{maximize } e_j^T R y$$

$$x^T R e_j \geq x^T C e_j \qquad e_j^T C y \geq e_j^T R y$$

$$\forall i, \quad x^T C e_j \geq x^T C e_i \qquad \forall i, \quad e_j^T R y \geq e_i^T R y$$

2. For the LP1, among all feasible choices of j find a solution (x, j) that maximizes $x^T C e_j$.

   Then, find the best-response strategy r of the row player to strategy j of the column player.

3. For the LP2, among all feasible choices of j find a solution (j, y) that maximizes $e_j^T R y$.

   Then, find the best-response strategy c of the column player to the strategy j of the row player.

4. If the maximum $x^T C e_j$ in Step 2 is greater or equal than the maximum $e_j^T R y$ in Step 3 then return strategy profile $(\frac{1}{2}x + \frac{1}{2}r, j)$; otherwise return strategy profile $(j, \frac{1}{2}y + \frac{1}{2}c)$.

This means that the maximum $x^T C e_j$ in Step 2 is greater or equal than u^*_{eg}. Furthermore, if $u^*_C \geq u^*_R = u^*_{eg}$, then by Lemma 5.5, there is a strategy profile (j, y^*) which is a feasible solution of the LP2, since $e_j^T C y^* \geq u^*_C \geq u^*_R = e_j^T R y^*$, and for all i, we have $e_j^T R y^* \geq e_i^T R y^*$.

This means that the maximum $e_j^T R y$ in Step 3 is greater or equal than u^*_{eg}. Therefore, the maximum of the maximum $x^T C e_j$ in Step 2 and the maximum $e_j^T Ry$ in Step 3 is greater or equal than u^*_{eg}.

Therefore, the strategy profile $(\frac{1}{2}x + \frac{1}{2}r, j)$ is a relative $\frac{1}{2}$-egalitarian additive $\frac{1}{2}$-NE, because:

$$\min\{(\frac{1}{2}x + \frac{1}{2}e_r)^T Re_j, (\frac{1}{2}x + \frac{1}{2}e_r)^T Ce_j\} \\ \geq \min\{x^T Ce_j, \frac{1}{2}x^T Ce_j\} = \frac{1}{2}x^T Ce_j \geq \frac{1}{2}u^*_{eg}.$$

The first inequality follows from $e_r^T Re_j \geq x^T Re_j \geq x^T Ce_j$ and the second one holds since $x^T Ce_j \geq u^*_{eg}$.

Finally, we observe that Algorithm 3 performs some basic polynomial-time computations and $2n$ calls to linear programming solver, and hence it runs in polynomial-time. □

5.3.2 The Second Case: $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$

We first describe a construction that yields an additive $\frac{1}{2}$-egalitarian additive ε-NE, and then we explain how this construction can be implemented in polynomial-time.

Let (x^*, y^*) be an egalitarian NE. Let u^*_R be the expected payoff for the row player and u^*_C be the expected payoff for the column player. We assume, without loss of generality, that $u^*_R \geq u^*_C$. We consider two cases, depending on whether $u^*_R \geq \frac{1-2\varepsilon}{1-\varepsilon}$ or not.

Case 1: $u^*_R \geq \frac{1-2\varepsilon}{1-\varepsilon}$. Let j be a pure best-response strategy of the column player to the strategy x^* of the row player such that $x^{*T} Re_j \geq u^*_R$ and $x^{*T} Ce_j = u^*_C$, whose existence follows by Lemma 5.5. Let r be a pure best-response strategy of the row player to the strategy j of the column player. Then, we claim that the strategy profile $(\frac{1}{2-u^*_R}x^* + (1 - \frac{1}{2-u^*_R})r, j)$ is an additive $\frac{1}{2}$-egalitarian additive ε-Nash equilibrium.

We first note that the fact that this strategy profile is an additive ε-Nash equilibrium can be shown in the same way as before.

Hence we focus on establishing the additive egalitarian approximation. Note that the payoff of the row player is bounded as follows:

$$\frac{1}{2-u_R^*}x^{*T}Re_j+(1-\frac{1}{2-u_R^*})e_r^TRe_j \geq \frac{1}{2-u_R^*}u_R^*+(1-\frac{1}{2-u_R^*})u_R^* \geq u_R^*.$$

This allows us to conclude the analysis because:

$$\begin{aligned}
&u_{eg}^* - \min\{(\frac{1}{2-u_R^*}x^* + (1-\frac{1}{2-u_R^*})e_r)^T Re_j,\\
&\qquad (\frac{1}{2-u_R^*}x^* + (1-\frac{1}{2-u_R^*})e_r)^T Ce_j\}\\
&\quad \leq u_{eg}^* - \min\{u_R^*, (\frac{1}{2-u_R^*}x^* + (1-\frac{1}{2-u_R^*})e_r)^T Ce_j\}\\
&\quad \leq u_{eg}^* - \min\{u_R^*, \frac{1}{2-u_R^*}x^{*T}Ce_j\} = u_{eg}^*\\
&\quad - \frac{1}{2-u_R^*}x^{*T}Ce_j = u_{eg}^* - \frac{1}{2-u_R^*}u_{eg}^*\\
&\quad \leq u_{eg}^* - \frac{1}{2}u_{eg}^* = \frac{1}{2}u_{eg}^* \leq \frac{1}{2}.
\end{aligned}$$

Case 2: $u_R^* < \frac{1-2\varepsilon}{1-\varepsilon}$. Find a strategy profile (x, y) such that for all i, we have $e_i{}^T Ry \leq u_R^*$ and $x^T Ce_i \leq u_R^*$. Existence of such a strategy profile follows from the definition of NE and from the fact that $u_R^* = \max\{x^{*T}Ry^*, x^{*T}Cy^*\}$. Next, we note that such a strategy profile is an additive $\frac{1-2\varepsilon}{1-\varepsilon}$-NE since $u_R^* < \frac{1-2\varepsilon}{1-\varepsilon}$. Therefore, since $\frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon$ for $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, such a strategy profile is an additive ε-NE. We conclude that it is an additive $\frac{1}{2}$-egalitarian additive ε-NE because for every $\varepsilon > \frac{1}{3}$, we have:

$$u_{eg}^* - \min\{x^T Ry, x^T Cy\} < \frac{1-2\varepsilon}{1-\varepsilon} < \frac{1}{2}.$$

Algorithm 4

1. For every j, solve the following two LPs:

$$
\begin{array}{llll}
LP1: & \text{maximize } x^T Ce_j & LP2: & \text{maximize } e_j^T Ry \\
 & x^T Re_j \geq x^T Ce_j & & e_j^T Cy \geq e_j^T Ry \\
\forall i, & x^T Ce_j \geq x^T Ce_i & \forall i, & e_j^T Ry \geq e_i^T Ry \\
\forall i, & x^T Re_j \geq e_i^T R\beta & \forall i, & e_j^T Cy \geq \alpha^T Ce_i
\end{array}
$$

2. For the LP1, among all feasible choices of j find a solution (x, j) that maximizes $x^T Ce_j$. For the strategy profile (x, j) assign $u_R = x^T Re_j$. Then, find the best-response strategy r of the row player to the strategy j of the column player.

3. For the LP2, among all feasible choices of j find a solution (j, y) that maximizes $e_j^T Ry$. For the strategy profile (j, y) assign $u_C = e_j^T Cy$. Then, find the best-response strategy c of the column player to strategy j of the row player.

4. If the maximum $x^T Ce_j$ in Step 2 is greater than the maximum $e_j^T Ry$ in Step 3, then if $u_R \geq \frac{1-2\varepsilon}{1-\varepsilon}$ return strategy profile $(\frac{1}{2-u_R}x + (1 - \frac{1}{2-u_R})r, j)$; else return strategy profile (x, β).

   Otherwise, if $u_C \geq \frac{1-2\varepsilon}{1-\varepsilon}$ then return strategy profile $(j, \frac{1}{2-u_C}y + (1 - \frac{1}{2-u_C})c)$; else return profile (α, y).

Theorem 5.13 *There is a polynomial-time algorithm that — given an ε, $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, and a bimatrix game (R, C) with values in $[0, 1]$ — computes an additive $\frac{1}{2}$-egalitarian additive ε-NE.*

Proof. We will show that if $\frac{3-\sqrt{5}}{2} \leq \varepsilon < \frac{1}{2}$, then Algorithm 4 returns an additive $\frac{1}{2}$-egalitarian additive ε-Nash equilibrium.

Let (x^*, y^*) be an egalitarian NE, and let u_R^* and u_C^* be the expected payoffs for the row player and the column player, respectively. If $u_R^* \geq u_C^* = u_{eg}^*$, then the definition of NE and Lemma 5.5 imply that there is a strategy profile (x^*, j) and a strategy β which are a feasible solution of LP1 since: $x^{*T} R e_j \geq u_R^* \geq u_C^* = x^{*T} C e_j$, and for all i, we have $x^{*T} C e_j \geq x^{*T} C e_i$ and $x^{*T} R e_j \geq u_R^* \geq e_i^T R \beta$. This means that the maximum $x^T C e_j$ in Step 2 is greater than or equal to u_{eg}^*.

Similarly, if $u_C^* \geq u_R^* = u_{eg}^*$, then the definition of NE together with Lemma 5.5 imply that there is a strategy profile (j, y^*) and a strategy α that are feasible solution of LP2, since: $e_j^T C y^* \geq u_C^* \geq u_R^* = e_j^T R y^*$, and for all i, we have $e_j^T R y^* \geq e_i^T R y^*$ and $e_j^T C y^* \geq u_C^* \geq \alpha^T C e_i$. This means that the maximum $e_j^T R y$ in Step 3 is greater than or equal to u_{eg}^*. As a result, the maximum of the maximum $x^T C e_j$ in Step 2 and the maximum $e_j^T R y$ in Step 3 is greater than or equal to u_{eg}^*.

Let us assume that the maximum $x^T C e_j$ in Step 2 is greater than the maximum $e_j^T R y$ in Step 3; the other symmetric case can be dealt with analogously. Therefore, if $u_R \geq \frac{1-2\varepsilon}{1-\varepsilon}$, then Algorithm 4 returns the strategy profile $(\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) r, j)$. Using the arguments analogous to those used earlier, we note that the strategy profile $(\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) r, j)$ is an additive ε-NE. Next, we argue that the strategy profile profile $(\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) r, j)$ is an additive $\frac{1}{2}$-egalitarian additive ε-NE, because:

$$u_{eg}^* - \min\{(\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) e_r)^T R e_j, (\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) e_r)^T C e_j\}$$
$$\leq u_{eg}^* - \min\{u_{eg}^*, (\frac{1}{2-u_R} x + (1 - \frac{1}{2-u_R}) e_r)^T C e_j\} \leq u_{eg}^*$$
$$- \min\{u_{eg}^*, \frac{1}{2-u_R} x^T C e_j\} \leq u_{eg}^* - \min\{u_{eg}^*, \frac{1}{2-u_R} u_{eg}^*\}$$
$$= u_{eg}^* - \frac{1}{2-u_R} u_{eg}^* \leq u_{eg}^* - \frac{1}{2} u_{eg}^* = \frac{1}{2} u_{eg}^* \leq \frac{1}{2}.$$

The first inequality holds since $(\frac{1}{2-u_R}x+(1-\frac{1}{2-u_R})e_r)^T Re_j \geq \frac{1}{2-u_R}x^T Ce_j + (1-\frac{1}{2-u_R})e_r^T Ce_j = x^T Ce_j \geq u^*_{eg}$, and the third inequality holds since $x^T Ce_j \geq u^*_{eg}$.

Next, we consider the remaining case, when $u_R < \frac{1-2\varepsilon}{1-\varepsilon}$, in which case Algorithm 4 returns the strategy profile (x, β). We first note that this strategy profile is an additive ε-NE since for every $\varepsilon \geq \frac{3-\sqrt{5}}{2}$ and for every i, we have: $e_i^T R\beta \leq u_R < \frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon$ and $x^T Ce_i \leq x^T Ce_j \leq u_R < \frac{1-2\varepsilon}{1-\varepsilon} \leq \varepsilon$.

Furthermore, the strategy profile (x, β) is an additive $\frac{1}{2}$-egalitarian additive ε-NE because for every $\varepsilon > \frac{1}{3}$, we have:

$$u^*_{eg} - \min\{x^T R\beta, x^T C\beta\} < \frac{1-2\varepsilon}{1-\varepsilon} < \frac{1}{2}.$$

In summary, we have shown that the strategy returned by Algorithm 4 is always an additive $\frac{1}{2}$-egalitarian additive ε-NE.

Finally, we observe that Algorithm 4 performs some basic polynomial-time computations and $2n$ calls to linear programming solver, and hence it runs in polynomial-time. $\square$

Part III

Communication complexity of approximate Nash equilibria

Chapter 6

Communication complexity

In this chapter, we study the approximation of Nash equilibria in bimatrix games in the context of *communication complexity*—amount of communication between the players needed to find an additive approximate Nash equilibrium. Ideas that are discussed in this chapter help to improve the approximation of Nash equilibria in the context of communication complexity (see [13]). We consider a natural scenario where there are two players who want to agree on a strategy profile that would be an additive approximate Nash equilibrium. The players are independent and each of them knows only her payoffs (as defined by the payoff matrices); but they can communicate to learn about the intentions and the entries of the payoff matrix of the other player. The objective of the players is to select mixed strategies, such that each player will have only limited incentives to deviate from their strategies (for example, that the strategy profile will form an additive ε-Nash equilibrium or an additive ε-well-supported Nash equilibrium).

The study of communication in the computation of equilibria in games has been frequently appearing in the game theory in the past, see, e.g., Conitzer and Sandholm [11], Hart and Mansour [33], and Goldberg and

Pastink [31] and the references therein. Very recently, Goldberg and Pastink [31] initiated the study of finding additive ε-Nash equilibria and additive ε-well-supported Nash equilibria with the goal of minimizing the parameter ε, while performing a small amount of communication between the players. It is easy to see that the task is trivial if communication of $\mathcal{O}(n^2)$ in the number n of strategies bits is allowed (assuming, for simplicity, that the precision of each entry is of constant length), each player can transmit its entire payoff matrix to the other player, and with this the players can compute an exact Nash equilibrium. Goldberg and Pastink have shown that even with no communication one can find an additive $\frac{3}{4}$-Nash equilibrium, but finding an additive $\frac{1}{2}$-Nash equilibrium is impossible without any communication [31]. Motivated by these positive and negative results, Goldberg and Pastink [31] focused on the most interesting setting, when one allows only a poly-logarithmic (in n) amount of communication between the players. They demonstrated that with so limited communication, one can compute additive 0.438-Nash equilibria and additive 0.732-well-supported Nash equilibria.

In this chapter, we show that with $\mathcal{O}(\log^2 n)$ bits of communication, one can find an additive $(\frac{3-\sqrt{5}}{2} + \delta)$-Nash equilibrium (note that $\frac{3-\sqrt{5}}{2} \approx 0.382$) and an additive $(\frac{2}{3} + \delta)$-well-supported Nash equilibrium, for any $\delta > 0$. This improves upon earlier bounds due to Goldberg and Pastink [31]. Furthermore, we note that the bounds we obtain are very close to the best known polynomial-time bounds with unlimited communication for these problems: our bound of additive 0.382-Nash equilibrium comes close to the bound of 0.3393 of Tsaknakis and Spirakis [51], and our bound of additive 0.667-well-supported Nash equilibrium comes close to the bound of 0.6528 [13].

Model of communication. We will consider the standard communication model, see, e.g., Kushilevitz and Nisan [40]. Each player has access to its payoff matrix, the row player can see the entire matrix R and the column player can see matrix C. The players can communicate between each other and they can perform local computations. The communication will be performed in rounds, and in each round each player may send a single bit of information to the other player. Between each round, each player can perform arbitrary local computations (though it should be noted that in our protocol, each player will perform only polynomial-time computations). At the end of the algorithm, each player computes her own mixed strategy.

We want to design a protocol that the players will use to collectively compute a strategy profile (x, y), which is an additive ε-Nash equilibrium or an additive ε-well-supported Nash equilibrium. The goal is to ensure that ε is as small as possible, and that the protocol performs only a *poly-logarithmic number of communication rounds*, that is, the players communicate only $\mathrm{polylog}(n)$ bits.

6.1 How to communicate mixed strategies

In this section, we will show how the players can communicate "approximations" of their mixed strategies with $\mathcal{O}(\log^2 n)$ bits of communication. Without loss of generality, we perform the analysis from the point of view of the row player.

Let y be a mixed strategy that the row player wants to communicate to the column player. Then, the row player can create an empirical distribution y_e from y by first taking $K = \lceil \ln n/\delta^2 \rceil$ random samples according to the distribution y and accordingly creating a multiset A of pure strategies in $\{1, \ldots, n\}$, for any $\delta > 0$. The probability of a pure strategy in the distribution y_e is its frequency in the multiset divided by the total number of samples K. Observe that by applying Hoeffding inequality, for any strategy k, it holds that:

$$\mathbf{Pr}\Big[\big|e_k^T R y_e - \mathbf{E}[e_k^T R y_e]\big| \geq \delta\Big] = \mathbf{Pr}\Big[\big|e_k^T R y_e - e_k^T R y\big| \geq \delta\Big] \leq 2e^{-2K\delta^2} \leq \frac{2}{n^2}.$$

It follows that,

$$\sum_{k=1}^{n} \mathbf{Pr}\Big[\big|e_k^T R y_e - \mathbf{E}[e_k^T R y_e]\big| \geq \delta\Big] = \sum_{k=1}^{n} \mathbf{Pr}\Big[\big|e_k^T R y_e - e_k^T R y\big| \geq \delta\Big] \leq \frac{2}{n}.$$

and hence,

$$\mathbf{Pr}\Big[\big|e_k^T R y_e - e_k^T R y\big| \leq \delta\Big] \geq 1 - \frac{2}{n}.$$

This means that with high probability the expected payoff of any strategy k when the row player picks strategy y_e differs from the expected payoff of strategy k when the row player picks strategy y by at most δ.

Note that there are $K = \mathcal{O}(\log n)$ strategies relevant to any y_e, and each such strategy can be described using $\mathcal{O}(\log n)$ bits. Therefore, any player can communicate to the other player an approximation y_e of its mixed strategy y using $\mathcal{O}(\log^2 n)$ bits so that, with high probability, the difference between the expected payoffs of these two strategies is upper bounded by δ. We conclude to the following lemma.

Lemma 6.1 *Let y be a mixed strategy, then a player can create an empirical mixed strategy y_e from y such that for any pure strategy k $\big|e_k^T R y_e - e_k^T R y\big| \leq \delta$, with high probability, for any $\delta > 0$.*

6.2 Communication-efficient additive 0.382-Nash equilibria

In this section we will describe how to achieve an additive $(\frac{3-\sqrt{5}}{2} + \delta)$-Nash equilibrium with poly-logarithmic communication between the players, for an arbitrary positive δ. (Since $\frac{3-\sqrt{5}}{2} \approx 0.38197$, this gives an additive 0.382-Nash equilibrium.) To do this, we will use the polynomial-time centralized algorithm—as it was described in Chapter 2 which gives an additive $\frac{3-\sqrt{5}}{2}$-Nash equilibrium, and we will show how to turn it into a protocol that uses only $\mathcal{O}(\log^2 n)$ bits of communication.

We will assume that the players only know their own payoff matrix. Therefore, the row player is able to solve on her own (without any communication with the column player) the zero-sum game $(R, -R)$, obtaining for it a Nash equilibrium (x^*, y^*) with value $v_R = (x^*)^T R y^*$, and the column player is able to solve on her own the zero-sum game $(-C, C)$, obtaining for it a Nash equilibrium $(\hat{x}, \hat{y})$ with value $v_C = (\hat{x})^T C \hat{y}$. We separate our analysis into two cases, one in which both values v_R, v_C are less than $\frac{3-\sqrt{5}}{2}$ and one in which at least one of the values is greater or equal to $\frac{3-\sqrt{5}}{2}$.

In the first case, when $\max\{v_R, v_C\} < \frac{3-\sqrt{5}}{2}$, the row player draws an empirical distribution y_e^* from y^* and the column player draws an empirical distribution $\hat{x}_e$ from $\hat{x}$. Then, they communicate the strategies y_e^*, $\hat{x}_e$, and play the strategy profile $(\hat{x}_e, y_e^*)$. This is an additive $(\frac{3-\sqrt{5}}{2} + \delta)$-Nash equilibrium, since for every pure strategy k, we have:

$$e_k^T R y_e^* \leq e_k^T R y^* + \delta \leq v_R + \delta \leq \tfrac{3-\sqrt{5}}{2} + \delta,$$

and

$$(\hat{x}_e)^T C e_k \leq (\hat{x})^T C e_k + \delta \leq v_C + \delta \leq v_R + \delta \leq \tfrac{3-\sqrt{5}}{2} + \delta.$$

In both cases, the first inequality holds because of Lemma 6.1. So, if r is a best-response strategy of the row player to the strategy y_e^* of the column player, we have:

$$e_r^T R y_e^* - (\hat{x_e})^T R y_e^* \leq \frac{3-\sqrt{5}}{2} + \delta,$$

since $(\hat{x_e})^T R y_e^* \geq 0$. Also, if c is a best-response strategy of the column player to the strategy $\hat{x_e}$ of the row player, we have:

$$(\hat{x_e})^T Ce_c - (\hat{x_e})^T Cy_e^* \leq \frac{3-\sqrt{5}}{2} + \delta,$$

since $(\hat{x_e})^T Cy_e^* \geq 0$.

In the second case, we assume without loss of generality that $v_R \geq v_C$. Then, the row player finds the empirical distribution x_e^* from x^*. We know that for every pure strategy k, we have:

$$(x_e^*)^T Re_k \geq (x)^T Re_k - \delta \geq v_R - \delta.$$

The first inequality holds because of Lemma 6.1 and the second one holds follows from the definition of the Nash equilibrium in zero-sum games. Then, the row player communicates the strategy x_e^* to the column player. The column player finds the best-response strategy j to this strategy and communicates it to the row player. Then, the row player finds the best-response strategy r to the strategy j of the column player. At the end, they play the strategy profile $(\frac{1}{2-v_R}x_e^* + \frac{1-v_R}{2-v_R}r, j)$. Similar to the proof of Theorem 2.10, this is an additive $(\frac{3-\sqrt{5}}{2} + \delta)$-Nash equilibrium, since the regret of the row player is

$$\frac{1-v_R+\delta}{2-v_R} \leq \frac{1-v_R}{2-v_R} + \delta \leq \frac{3-\sqrt{5}}{2} + \delta.$$

On the other hand, the regret of the column player is $\frac{1-v_R}{2-v_R} \leq \frac{3-\sqrt{5}}{2}$. Notice that since $\frac{3-\sqrt{5}}{2} < 0.382$, we can conclude that one can find an additive $(0.382+\delta)$-Nash equilibrium with communication complexity $\mathcal{O}(\log^2 n)$ bits, for any $\delta > 0$.

6.3 Communication-efficient additive $\frac{2}{3}$-well-supported Nash equilibria

Similarly as in the case of additive ε-Nash equilibria, in the case of additive ε-well-supported Nash equilibria we will describe a centralized algorithm to achieve an additive $\frac{2}{3}$-well-supported Nash equilibrium, and then we will describe how we can transform it in a communication efficient decentralized algorithm to achieve an additive $(\frac{2}{3} + \delta)$-well-supported Nash equilibrium. This centralized algorithm inspired the work of the paper [13].

Theorem 6.2 *For any bimatrix game with payoff matrices $(R, C) \in [0, 1]^{n\times n}$, there is a polynomial-time algorithm that returns an additive $\frac{2}{3}$-well-supported Nash equilibrium.*

Proof. Let $(R, -R)$ and $(-C, C)$ be two zero-sum games with values $v_R = (x^*)^T Ry^*$, $v_C = (\hat{x})^T C\hat{y}$ and with Nash equilibria (x^*, y^*) and $(\hat{x}, \hat{y})$, respectively. We assume without loss of generality that $v_R \geq v_C$. Then, the following Algorithm 1 returns an additive $\frac{2}{3}$-well-supported Nash equilibrium.

In Step 1, the algorithm returns the strategy profile $(\hat{x}, y^*)$, this is an additive $\frac{2}{3}$-well-supported Nash equilibrium since for every strategy k of the row player, we have:

$$e_k^T Ry^* \leq v_R \leq \tfrac{2}{3},$$

and for every column player pure strategy k, we have:

$$(\hat{x})^T Ce_k \leq v_C \leq v_R \leq \tfrac{2}{3}.$$

In Step 2.(a), this is also an additive $\frac{2}{3}$-well-supported Nash equilibrium since the row player plays the best-response strategy and for every column player strategy k, we have $(x^*)^T Ce_k \leq \frac{2}{3}$.

Algorithm 1

1. If $v_R \leq \frac{2}{3}$, then **return** $(\hat{x}, y^*)$.
2. Otherwise
   (a) If for all j it holds that $(x^*)^T Ce_j \leq \frac{2}{3}$, then **return** (x^*, y^*).
   (b) Otherwise
      i. Find a column c such that $(x^*)^T Ce_c > \frac{2}{3}$.
      ii. In this column, find is a pure strategy profile (i, c) such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$.
      iii. **Return** pure strategy profile (i, c).

In Step 2.(b), for every column player strategy k, $v_R = (x^*)^T Re_k > \frac{2}{3}$, so there is at least one column c such that $(x^*)^T Re_c > \frac{2}{3}$ and $(x^*)^T Ce_c > \frac{2}{3}$. In this column, there is at least one row i such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$. This is an additive $\frac{2}{3}$-well-supported Nash equilibrium.

For the sake of contradiction, assume that there is no such pure strategy profile. Thus, for every row player strategy i, $R_{ic} \geq \frac{1}{3}$ implies that $C_{ic} < \frac{1}{3}$. Then, let p be the minimum total probability of the entries such that $R_{ic} \geq \frac{1}{3}$, we can bound this probability

$$p + (1-p)\left(\frac{1}{3} - t\right) > \frac{2}{3},$$

for any arbitrarily small $t > 0$ and hence we obtain

$$p > \frac{\frac{1}{3} + t}{\frac{2}{3} + t}.$$

But, the maximum payoff of the column player is

$$p\left(\frac{1}{3} - t\right) + 1 - p = 1 - p\left(\frac{2}{3} + t\right) \leq \frac{2}{3} - t < \frac{2}{3},$$

for any $p > \frac{\frac{1}{3}+t}{\frac{2}{3}+t}$. This is a contradiction, so there is at least one pure strategy profile (i, c) such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$. $\square$

We now describe how to transform the previous algorithm in the context of *communication complexity*. First of all, the row player solves the zero-sum game $(R, -R)$ with equilibrium (x^*, y^*) and value $v_R = (x^*)^T R y^*$, and the column player solves the zero-sum game $(-C, C)$ with equilibrium $(\hat{x}, \hat{y})$ and value $v_C = (\hat{x})^T C \hat{y}$. We assume, without loss of generality, that $v_R \geq v_C$. We separate the analysis in to two cases: $v_R < \frac{2}{3}$ and $v_R \geq \frac{2}{3}$.

If $v_R < \frac{2}{3}$, then the row player finds an empirical distribution y_e^* from y^* and the column player finds the empirical distribution $\hat{x}_e$ from $\hat{x}$. The players send each other their empirical distributions and then they play the strategy profile $(\hat{x}_e, y_e^*)$. This is an additive $(\frac{2}{3} + \delta)$-well-supported Nash equilibrium since for every pure strategy k, we have:

$$e_k^T R y_e^* \leq e_k^T R y^* + \delta \leq v_R + \delta < \tfrac{2}{3} + \delta,$$

$$\hat{x}_e^T C e_k \leq \hat{x}^T C e_k + \delta \leq v_C + \delta \leq v_R + \delta < \tfrac{2}{3} + \delta.$$

In both cases, the first inequality holds because of Lemma 6.1.

If $v_R \geq \frac{2}{3}$, and if $(x^*)^T C e_k \leq 2/3$ for all k, by Lemma 6.1 we have that $(x_e^*)^T C e_k \leq \frac{2}{3} + \delta$ for all k, then the row player sends the empirical distribution y_e^* to the column player and the players play (x_e^*, y_e^*). This is an additive $(\frac{2}{3} + \delta)$-well-supported Nash equilibrium, since for every pure strategy k, we have:

$$(x_e^*)^T C e_k \leq \frac{2}{3} + \delta.$$

Also, for every $k \in \text{supp}(x_e^*)$, we have

$$e_k^T R y_e^* \geq e_k^T R y^* - \delta = v_R - \delta \geq \frac{2}{3} - \delta,$$

The first inequality holds by Lemma 6.1 and the equality holds by the definition of Nash equilibria and the fact that in the support of (x_e^*) belong strategies that they are best-response strategies with expected payoff v_R. Therefore, the incentive to deviate to the best-response strategy is at most

$$1 - e_k^T R y_e^* \leq \frac{1}{3} + \delta < \frac{2}{3} + \delta.$$

Otherwise there is a column c such that $(x_e^*)^T C e_c > \frac{2}{3} + \delta$ and $(x_e^*)^T R e_c \geq v_R - \delta \geq \frac{2}{3} - \delta$. We claim that for some $i \in \mathrm{supp}(x_e^*)$, there is a pure strategy profile such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$.

For the sake of contradiction, let assume that there is no such a pure strategy profile. Thus, for every pure strategy $i \in \mathrm{supp}(x_e^*)$, $R_{ic} \geq \frac{1}{3}$ implies that $C_{ic} < \frac{1}{3}$. Then, let p be the minimum total probability of the entries such that $R_{ic} \geq \frac{1}{3}$, for every pure strategy $i \in \mathrm{supp}(x_e^*)$, we can bound this probability as

$$p + (1-p)\left(\frac{1}{3} - t\right) \geq \frac{2}{3} - \delta,$$

for any arbitrary small $t > 0$ and therefore we have

$$p \geq \frac{\frac{1}{3} - \delta + t}{\frac{2}{3} + t}.$$

The maximum payoff of the column player is

$$p\left(\frac{1}{3} - t\right) + 1 - p = 1 - p\left(\frac{2}{3} + t\right) \leq \frac{2}{3} + \delta - t < \frac{2}{3} + \delta,$$

for any $p \geq \frac{\frac{1}{3}-\delta+t}{\frac{2}{3}+t}$. This is a contradiction, so there is at least one pure strategy profile (i, c) such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$.

So, the row player sends the empirical distribution x_e^* to the column player, the column player finds the best-response strategy c to the strategy x_e^*. Then, the players, via communication, can find a pure strategy (i, c), for $i \in supp(x_e^*)$, such that $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$ checking at most $\mathcal{O}(\log n/\delta^2)$ pure strategies, since the size of the support of x_e^* is $\mathcal{O}(\log n/\delta^2)$. Then, the players play the strategy profile (i, c), but since $R_{ic} \geq \frac{1}{3}$ and $C_{ic} \geq \frac{1}{3}$, this is an additive $\frac{2}{3}$-well-supported Nash equilibrium.

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